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Cancellation and Direct Summands in Dimension 1

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Let Λ be a module-finite algebra over a commutative noetherian ring of Krull dimension 1. We extend Roiter's direct-summand theorem to arbitrary finitely generated Λ-modules, obtaining a sharpened form of Serre's direct-summand theorem in this setting. We also extend Drozd's cancellation theorem to arbitrary finitely generated Λ-modules, obtaining a sharpened form of Bass's cancellation theorem in this setting. A corollary is that, over commutative reduced noetherian rings of dimension 1, direct-sum cancellation holds in every genus of finitely generated modules. (This becomes false if the ring has nilpotent elements.) Another corollary is that if direct-sum cancellation holds in the genera of Λ-modules M and N, then it holds in the genus of $M \oplus N$. This seems to be new, even for the modules that occur in integral representation theory. The main thrust of this paper is to close the gap between integral representation theory and the rest of module theory by eliminating hypotheses concerning the existence of maximal orders (of "finite normalization," in the commutative case) and allowing our rings to have nilpotent ideals.

Throughout this paper Λ denotes a module-finite algebra over a commutative noetherian ring R of Krull dimension $\leqslant 1$, and "Λ-module" means "finitely generated left Λ-module," unless the contrary is explicitly stated.

In integral representation theory, one studies Λ-modules M, assuming a number of standard hypotheses: R is a Dedekind domain, Λ is an R-order in a semisimple artinian ring A, M is torsion-free, and Λ is contained in a maximal R-order in A. The main purpose of the present paper is

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to eliminate these hypotheses, obtaining theorems that are both more general and more ring-theoretically natural. Our main contribution lies in eliminating the hypothesis that Λ is contained in a maximal R-order in A, and in combining this with already-developed techniques for deleting the remaining hypotheses.

Assume for the moment that Λ is either one of the orders studied in integral representation theory, or else that Λ is the coordinate ring of an affine curve and $R = \Lambda$. Then $\Lambda_{\mathfrak{m}}$ is a maximal $R_{\mathfrak{m}}$-order (in the commutative case, a discrete valuation ring) for almost all maximal ideals $\mathfrak{m}$ of R. This fundamental fact dominates the theory of finitely generated torsion-free Λ-modules M because of its consequences that, whenever $\Lambda_{\mathfrak{m}}$ is a maximal order, $M_{\mathfrak{m}}$ is $\Lambda_{\mathfrak{m}}$-projective, and its isomorphism class is determined by its rank.

On the other hand, M. Hochster [H] has given an example of a commutative noetherian domain Λ of Krull dimension 1 with infinitely many maximal ideals $\mathfrak{m}$, such that *no* $\Lambda_{\mathfrak{m}}$ is a discrete valuation ring. In fact $\Lambda_{\mathfrak{m}}$ is always the localization of $K[x^2, x^3]$ (K an algebraically closed field) at its maximal ideal $\langle x^2, x^3 \rangle$.

The clue to dealing with this situation is the observation that, even over Hochster's ring, every finitely generated module becomes projective when localized at almost all maximal ideals. But, because Λ has no localizations that are discrete valuation rings, the finite set of exceptional maximal ideals varies from module to module.

Now return to a general module-finite algebra Λ over a commutative noetherian ring R of dimension $\leqslant 1$. The outline of this paper is as follows.

Section 1. *Fixed Notation*; *Orders, Lattices*; *Genus.* This section gives our definitions of these familiar terms in suitable generality, and briefly summarizes our reduction of Λ-module structure to the situation that Λ is an R-order in a semisimple artinian ring, and the Λ-module being studied is a Λ-lattice. This situation occurs as the crux of most of our proofs.

The localization $R_Q = \bigoplus_{\mathfrak{p}} \{R_{\mathfrak{p}} \mid \mathfrak{p}$ is a minimal prime ideal of $R\}$, in our setting, plays the role traditionally played by the quotient field of R, when R is an integral domain (and reduces to this when R is an integral domain).

Section 2. *Almost Always Projective, Normalization.* Let M be a Λ-module. Our starting point is the following simple observation: If M_Q is Λ_Q-projective, that is, if $M_{\mathfrak{p}}$ is $\Lambda_{\mathfrak{p}}$-projective for all minimal prime ideals $\mathfrak{p}$ of R, then $M_{\mathfrak{m}}$ is $\Lambda_{\mathfrak{m}}$-projective for *almost* all maximal maximal ideals $\mathfrak{m}$ of R, that is, for all but finitely many $\mathfrak{m}$. We abbreviate this by saying that $M_{\mathfrak{m}}$ is projective (a$\forall\mathfrak{m}$).

In the special case that Λ_Q is a semisimple artinian ring, this hypothesis on M_Q is satisfied for all Λ-modules M. Suppose, in addition, that Λ is an R-order in Λ_Q (as defined in Section 1). We conclude that, for every *faithful* Λ-module M, $M_\mathfrak{m}$ is a $\Lambda_\mathfrak{m}$-progenerator (a$\forall\mathfrak{m}$). A consequence of this is that $\Lambda_\mathfrak{m}$ is a direct product of full matrix rings over (noncommutative) integral domains (a $\forall\mathfrak{m}$).

The second idea in this section assumes that Λ is an R-order in the semisimple artinian ring Λ_Q, and describes how to deal with the situation that Λ is not contained in a maximal R-order in Λ_Q. In this situation Λ still has a *normalization*, that is, a maximal element Γ of the set of rings between Λ and Λ_Q consisting of elements integral over $R \cdot 1_\Lambda$. Of course Γ need not be a finitely generated R-module, even in the commutative case. However, given any Λ-lattice M, there is a ring Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) that is finitely generated as an R-module and such that ΩM is a projective Ω-module. Much of the role classically played by maximal orders, when they exist, can be played by Ω.

Section 3. *Units Action, Connection with* $\mathbf{K}_1$. Again consider the situation in which Λ is an R-order in the semisimple artinian ring Λ_Q. Let M be a Λ-lattice, and choose a ring Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) such that ΩM is Ω-projective. Our objective is to describe all isomorphism classes of Λ-lattices N in the (Λ, Ω)-*genus* of M (i.e., all N locally isomorphic to M such that $\Omega N \cong \Omega M$) in terms of an action of the group of units of an artinian ring Ω/T on M. All of our later results make use of this.

This units action was previously exploited in the commutative case by Wiegand [W, WW], assuming that the normalization of R is a finitely generated R-module, then, without the finiteness assumption, by Rush [Ru]. A noncommutative version of this units action was used by Guralnick [G3] and Swan [S2], simplifying an earlier idea of Fröhlich [F]. These noncommutative sources assume that Λ is contained in a maximal order Ω. The paper [G3] actually used an action of the units of a localization Ω_π, rather than Ω/T, and Wiegand's use of determinants, in the commutative case, was replaced by the use of $\mathbf{K}_1$.

In our original work on the problems considered in this paper, we used an adaptation of the action of Ω_π. However, C. Odenthal suggested that the conceptually simpler use, in the commutative situation, of the artinian ring Ω/T could be adapted to our noncommutative needs, and the connection with $\mathbf{K}_1$ could be made just as easily, too. This is the scheme we actually use.

Section 4. *Local versus Global Direct Summands.* Return to the general Λ, as at the beginning of this paper. We show that if a Λ-module M has a decomposition $M = \bigoplus_{i=1}^n M_i$ then any Λ-module N locally isomorphic to M has a decomposition $N = \bigoplus_{i=1}^n N_i$ with each N_i locally isomorphic to M_i. This is well known for lattices over the orders that occur in integral representation theory [CR1, (31.13)], and seems to have first been noticed by Jacobinski [J] for the case that R is the ring of integers in an algebraic number field.

Section 5. *Cancellation in a Genus.* This section establishes our first main results. We say that "cancellation holds in the genus of M," for a Λ-module M, if

$$(0.1) \qquad L \oplus X \cong N \oplus X \Rightarrow L \cong N$$

whenever L, X, N belong to the genus of M (i.e., $M_{\mathfrak{m}} \cong L_{\mathfrak{m}} \cong X_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R). Cancellation in the genus of M implies that (0.1) holds under the more familiar, and more general, hypothesis that X is a direct summand of a direct sum of copies of L, and L is in the genus of M. (See Remark 5.15.) But we prefer to work with the conceptually simpler and more easily remembered notion of cancellation in a genus.

Our main cancellation theorem gives a sufficient condition for cancellation to hold in the genus of M. The condition is that certain division R_Q-algebras associated with M be "universally stabilizing," a term that we define in Section 5. In order to state which division algebras are required to have this property, let $E(V)$ denote the endomorphism ring of a module V.

Let M be a Λ-module, and note that Λ_Q is an artinian ring. Suppose, for every indecomposable Λ_Q-module V whose isomorphism class occurs exactly once when M_Q is written as a direct sum of indecomposable modules, that the division algebra $E(V)/\mathrm{rad}(E(V))$ is universally stabilizing. We prove that cancellation then holds in the genus of M.

We prove that every field is universally stabilizing. This generalizes the cancellation theorem of Drozd mentioned in the abstract of this paper; and we discuss this in more detail in Section 5.

In the special case that Λ is commutative and has no nilpotent elements, we deduce that cancellation holds in every genus of Λ-modules. It is interesting to observe that when Λ has nilpotent elements this cancellation fails, as shown in [GLW].

For another interesting special case, note that our sufficient condition is trivially satisfied for any (say, indecomposable) Λ-module M that is "large enough" so that every isomorphism class of indecomposable direct summand of M_Q occurs at least twice in a decomposition of M_Q.

Finally, we note that, when R is the ring of algebraic integers in any global field, all division algebras satisfying the Eichler condition also satisfy our "universally stabilizing" condition. (But we do not give a new proof of this.)

To compare our result with Bass's cancellation theorem, for the rings we are considering, suppose in (0.1) that L, N, X are projective and for every maximal ideal $\mathfrak{m}$ of R, that $\Lambda_{\mathfrak{m}}$ is isomorphic to a direct summand of $L_{\mathfrak{m}}$. Then Bass's sufficient condition for cancellation (0.1) is that $(\Lambda_Q)^2$ is isomorphic to a direct summand of L_Q. This condition is not satisfied if L is in the genus of Λ. Our sufficient condition is satisfied if every *indecomposable summand* of L_Q occurs at least twice in L_Q (or, more generally, has a universally stabilizing associated division algebra, if it occurs exactly once). This can be satisfied, for example, even if L itself is indecomposable. These assertions follow from Remark 5.15, with $M = \Lambda$.

Section 6. *Roiter's Theorem.* Let X and V be Λ-modules. We wish to conclude, from local data, that $X \mid V$ (X is isomorphic to a direct summand of V).

One sufficient condition that we establish ("Jacobinski's theorem") is that $X_{\mathfrak{m}} \mid V_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R and every indecomposable direct summand of X_Q occurs more often as a direct summand of V_Q. A special case of this is the sufficient condition ("Roiter's Theorem") that $V = M \oplus N$ where $X_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R and every *indecomposable* direct summand of X_Q is isomorphic to a direct summand of N_Q. This condition on N_Q is automatically satisfied if, as in Roiter's original theorem, Λ_Q is semisimple artinian and the Λ-module N is faithful.

1. Fixed Notation; Orders, Lattices; Genus

Throughout this paper, *module* means "finitely generated left module," unless the contrary is stated.

1.1. *Fixed Notation* (R, Λ, Q). Throughout this paper R denotes a commutative noetherian ring of Krull dimension $\leqslant 1$, and Λ denotes a module-finite R-algebra. We do not suppose that $R \subseteq \Lambda$.

Q denotes the finite set of minimal (=height zero) prime ideals of R. Then R_Q denotes the localization

$$R_Q = \left\{ r/s \;\middle|\; r, s \in R,\ s \notin \bigcup \{\mathfrak{p} \in Q\} \right\} = \bigoplus_{\mathfrak{p} \in Q} R_{\mathfrak{p}}. \tag{1.1.1}$$

This is an artinian ring, since it is noetherian and all of its prime ideals are maximal.

Note that, if R has no nilpotent elements, then $R - \bigcup Q$ is the set of *regular elements* (=nonzero divisors) of R, hence R_Q *is the total quotient ring of* R, and each $R_{\mathfrak{p}}$ in (1.1.1) is a field. If R has nilpotent elements, however, the kernel of the canonical map $R \to R_Q$ can be nonzero. For example $R = \mathbb{Z}[x]/\langle x^2, 5x\rangle = \{a + b\bar{x} \mid a \in \mathbb{Z},\ b \in \mathbb{Z}/5\mathbb{Z}, \text{ and } \bar{x}^2 = 0\}$ yields $R_Q = \mathbb{Q}$.

Since R_Q is an artinian ring, so is Λ_Q.

1.2. Definitions (R-order, Λ-lattice). Suppose that Λ_Q is a semisimple artinian ring and $\Lambda \subseteq \Lambda_Q$ (i.e., the canonical map $\Lambda \to \Lambda_Q$ is a monomorphism). In this situation we say that Λ *is an R-order in the semisimple artinian ring* Λ_Q.

By a *Λ-lattice M* we mean a (finitely generated) Λ-submodule of some free Λ-module. Equivalently (since every Λ_Q-module is projective), a Λ-module M is a Λ-lattice if and only if M is a (finitely generated) Λ-submodule of some Λ_Q-module. We only use the terms "order" and "lattice" when Λ_Q is a semisimple artinian ring.

In the situation studied in integral representation theory, where R is a Dedekind domain and Λ is finitely generated and torsion-free as an R-module, it is easily verified that these definitions of "order" and "lattice" coincide with the usual ones.

Note that, when Λ is an R-order in the semisimple artinian ring $A = \Lambda_Q$, the ring A is the Goldie quotient ring of Λ, and is therefore independent of the particular commutative ring R over which Λ is a module-finite R-algebra.

1.3. *Genus.* Return to the general situation in Notation 1.1.

For Λ-modules M, N we say, "N is in the *genus* of M" and write $N \in \operatorname{gen}(M)$ if $M_\mathfrak{m} \cong N_\mathfrak{m}$ as $\Lambda_\mathfrak{m}$-modules for every maximal ideal $\mathfrak{m}$ of R.

The statement "N is in the genus of M" is independent of the particular commutative ring R over which Λ is a module-finite algebra.

In proving this we can suppose that $R \subseteq \Lambda$. It then suffices to prove that M and N are in the same genus with respect to R if and only if they are in the same genus with respect to the center S of Λ. This, in turn, follows from the facts that (i) S is integral over R; and (ii) if $R/\operatorname{rad}(R)$ is zero-dimensional, then $N \in \operatorname{gen}(M)$ if and only if $N \cong M$ [GW] or [EG].

The following lemma reduces the problem of cancellation in $\operatorname{gen}(M)$ to the case that Λ is an order in a semisimple artinian ring and $M = \Lambda$. The reductions involved are fairly well known, but we include a precise statement and partial proof because the lemma is crucial to almost everything that follows.

1.4. Lemma. *For every (module-finite) R-algebra Λ and left Λ-module M there is an R-order Λ' in a semisimple artinian ring $(\Lambda')_Q$ and an additive functor Ψ:* $\operatorname{mod}(\Lambda) \to \operatorname{mod}(\Lambda')$ *such that*

(i) *Ψ provides a bijection between the set of isomorphism classes of Λ-modules in* $\operatorname{gen}(M)$ *and the set of isomorphism classes of (necessarily projective) left Λ'-modules in* $\operatorname{gen}(\Lambda')$;

(ii) *Λ' has no nonzero R-submodules of finite length and no* 1*-sided ideals of finite Λ'-length; and*

(iii) $\Psi(M) = \Lambda'$.

Proof. Let $E(\ldots)$ denote "endomorphism ring of $(\ldots)$." The functor Ψ is the composition of functors Φ_1, Φ_2, and Φ_3, which we proceed to describe.

(1.4.1) The additive functor $\Phi_1(N) = \hom(M, N)$ provides a bijection between the isomorphism classes of Λ-modules $N \in \operatorname{gen}(M)$ and the isomorphism classes of (necessarily projective) left $E(M)$-modules in the genus of $E = E(M)$.

(1.4.2) The additive functor $\Phi_2(U) = U/((\text{nilrad } E) \cdot U)$ provides a bijection between the isomorphism classes of left E-modules in $\operatorname{gen}(E)$ and the isomorphism classes of left $E/(\text{nilrad } E)$-modules in the genus of the semiprime ring $\bar{E} = E/(\text{nilrad } E)$.

(1.4.3) $\bar{E} = \Lambda' \oplus G$ (ring direct sum) where Λ' is an R-order in the semisimple artinian ring $(\Lambda')_Q$, G has finite length as an R-module, Λ' has no nonzero R-submodules of finite length, and Λ' has no left or right ideals of finite Λ'-length.

(1.4.4) The additive functor $\Phi_3(V) = \Lambda' \cdot V$ provides a bijection between the isomorphism classes of left $\bar{E}$-modules in the genus of $\bar{E}$ and the isomorphism classes of left Λ'-modules in the genus of Λ'.

The functor Φ_1 is discussed in more detail [GL, 1.1]. The functor Φ_2 is discussed in [G1, 3.3] (but correct a misprint by replacing "nil" by "nilpotent").

Since $\bar{E}$ is a noetherian semiprime ring, its left socle G equals its right socle, and is generated by a central idempotent [CRo]. Let Λ' be the 2-sided ideal such that $\bar{E} = \Lambda' \oplus G$. Since G is a finitely generated module over its central subring $R \cdot 1_G$, the ring $R \cdot 1_G$ is artinian [E], and hence G has finite length as an R-module.

To see that Λ' has no nonzero R-submodules of finite length, suppose that Rx $(x \in \Lambda')$ is such a submodule. Since the R-module Λ' is finitely generated, the R-module $\Lambda' x$ is a homomorphic image of the direct sum of a finite number of copies of Rx. So $\Lambda' x$ has finite length as an R-module, hence as a Λ'-module; hence it intersects the socle G of $\bar{E}$, a contradiction.

To prove (1.4.4) it suffices to show that every genus of G-modules consists of a single isomorphism class. But since G has finite length as an

R-module, $G = \bigoplus \{G_{\mathfrak{m}} \mid \mathfrak{m}$ is a maximal ideal of $R\}$. (Only finitely many of these localizations are nonzero.) ∎

Note that if Λ is artinian, in Lemma 1.4, we get $\Lambda' = 0$. This causes no difficulty since every genus of Λ-modules consists of a single isomorphism class.

1.5. *Remark.* To enhance the analogy between the R-order Λ' in a semisimple artinian ring and the situation encountered in integral representation theory (where R is a Dedekind domain, but not a field), we note the following:

(1.5.1) No minimal prime ideal of the image of R in Λ' is a maximal ideal.

The proof is the same as that in [GL, paragraph above 2.2].

Finally, we reduce the problem of local versus global summands of modules to the case of orders in semisimple artinian rings. The notation $\operatorname{div}(M)$ refers to the category of all direct summands of the modules M^n ($n \geqslant 1$), and $L \mid N$ means "L is isomorphic to a direct summand of N."

1.6. LEMMA. *Let Λ, M, Λ', Ψ be as in Lemma* 1.4, *and let N be a Λ-module.* (*The letter* $\mathfrak{m}$ *is used to denote maximal ideals of* R.) *Then*

(i) *If $N_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ for every* $\mathfrak{m}$, *then* $N \in \operatorname{div}(M)$.

(ii) Ψ *maps* $\operatorname{div}(M)$ *onto* $\operatorname{div}(\Lambda')$.

(iii) *For L, N in* $\operatorname{div}(M)$, $L \mid N$ *if and only if* $L_{\mathfrak{m}} \mid N_{\mathfrak{m}}$ *for every* $\mathfrak{m}$ *and* $\Psi(L) \mid \Psi(N)$.

(iv) *If $(\Lambda_{\mathfrak{m}})'$ and $\Psi_{\mathfrak{m}}$ are computed from the $R_{\mathfrak{m}}$-algebra $\Lambda_{\mathfrak{m}}$ and module $M_{\mathfrak{m}}$ the same way that Λ' and Ψ are computed from Λ and M, then* $(\Lambda_{\mathfrak{m}})' = (\Lambda')_{\mathfrak{m}}$ *and* $\Psi_{\mathfrak{m}}(N_{\mathfrak{m}}) = (\Psi(M))_{\mathfrak{m}}$.

Proof. For (i) see [G1, 3.1]. It suffices to prove (ii) and (iii) separately for the functors Φ_i whose composition is Ψ. For Φ_1 and Φ_2 see, for example, [G1, 3.2 and 3.3]. The case of Φ_3 is obvious. The claimed localizations in (iv) hold because all modules involved are noetherian. ∎

2. ALMOST ALWAYS PROJECTIVE, NORMALIZATION

As mentioned in the introduction, the abbreviation $(\mathrm{a}\forall \cdots)$ means "for almost all $\cdots$." In this section, $\mathfrak{m}$ always denotes a maximal ideal of R.

2.1. LEMMA. *Let M and N be Λ-modules.*

(i) *If* $M_Q \cong N_Q$ *then* $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ (a$\forall\mathfrak{m}$).

(ii) *If there is a* Λ_Q*-surjection* $M_Q \twoheadrightarrow N_Q$ *then some* $\alpha \in \hom_\Lambda(M, N)$ *becomes a* $\Lambda_{\mathfrak{m}}$*-surjection* $M_{\mathfrak{m}} \twoheadrightarrow N_{\mathfrak{m}}$ (a$\forall\mathfrak{m}$).

(iii) *If there is a split* Λ_Q*-surjection* $M_Q \twoheadrightarrow N_Q$ *then some* $\alpha \in \hom_\Lambda(M, N)$ *becomes a split* $\Lambda_{\mathfrak{m}}$*-surjection* $M_{\mathfrak{m}} \twoheadrightarrow N_{\mathfrak{m}}$ (a$\forall\mathfrak{m}$).

Proof. (i) We can choose a pair of mutually inverse isomorphisms α/s and β/s, where $\alpha \in \hom(M, N)$, $\beta \in \hom(N, M)$, and $s \in R - \bigcup Q$, since M and N are finitely presented. Then there exists $t \in R - \bigcup Q$ such that $t\alpha\beta = ts^2 \cdot 1_M$ and $t\beta\alpha = ts^2 \cdot 1_N$. Since $s, t \in R - \bigcup Q$ and R is noetherian of dimension $\leqslant 1$, the elements s and t belong to only finitely many maximal ideals $\mathfrak{m}$ of R. In any other $R_{\mathfrak{m}}$ they become units, hence $\alpha/1$ and $\beta/1$ become isomorphisms. This proves (i).

The proofs of (ii) and (iii) are similar, except that we start with a surjection (α/s): $M_Q \twoheadrightarrow N_Q$. ∎

2.2. Theorem. (i) *Let* M *be a* Λ*-module such that* M_Q *is* Λ_Q*-projective. Then* $M_{\mathfrak{m}}$ *is* $\Lambda_{\mathfrak{m}}$*-projective* (a$\forall\mathfrak{m}$).

(ii) *Suppose that* M_Q *is a* Λ_Q*-progenerator. Then* $M_{\mathfrak{m}}$ *is a* $\Lambda_{\mathfrak{m}}$*-progenerator* (a$\forall\mathfrak{m}$).

Proof. (i) By hypothesis there is a split surjection $F_Q \twoheadrightarrow M_Q$ for some free Λ-module F. So, by (iii) of the lemma, there is a split surjection $F_{\mathfrak{m}} \twoheadrightarrow M_{\mathfrak{m}}$ (a$\forall\mathfrak{m}$), as desired.

(ii) By (i), $M_{\mathfrak{m}}$ is projective (a $\forall\mathfrak{m}$). By hypothesis there is a surjection $(M_Q)^n \twoheadrightarrow R_Q$ for some n. Again, by the lemma, there is a surjection $(M_{\mathfrak{m}})^n \twoheadrightarrow R_{\mathfrak{m}}$ (a$\forall\mathfrak{m}$). ∎

2.3. Corollary. *Let* Λ *be an* R*-order in the semisimple artinian ring* Λ_Q, *and* M *a* Λ*-module. Then*

(i) $\Lambda_{\mathfrak{m}}$ *is a direct sum of full matrix rings over (possibly non-commutative) integral domains* (a$\forall\mathfrak{m}$); *and*

(ii) $M_{\mathfrak{m}}$ *is* $\Lambda_{\mathfrak{m}}$*-projective* (a$\forall\mathfrak{m}$). *If* M *is faithful, then* $M_{\mathfrak{m}}$ *is a* $\Lambda_{\mathfrak{m}}$*-progenerator* (a$\forall\mathfrak{m}$).

Proof. (i) Obviously Λ_Q contains an R-order Ω that is a direct sum of full matrix rings over R-orders in the division rings of Λ_Q. Since $\Lambda_Q = \Omega_Q$ we have $\Lambda_{\mathfrak{m}} = \Omega_{\mathfrak{m}}$ (a$\forall\mathfrak{m}$) by the proof of Lemma 2.1(i).

(ii) Since Λ_Q is semisimple artinian, every Λ_Q-module is projective; and is a progenerator if and only if it is faithful. Now apply the theorem. ∎

2.4. DEFINITIONS (*Γ, $\tilde{R}$, normalization*). Let Λ be an R-order in the semisimple artinian ring Λ_Q, and suppose that $R \subseteq \Lambda$.

Then R has no nilpotent elements $\neq 0$ (because any such element would generate a nilpotent ideal of Λ_Q). As observed in Section 1, R_Q is then the total quotient ring of R.

We reserve the notation Γ for a *normalization* of Λ, that is, for a maximal element of the set of rings Γ' such that $\Lambda \subseteq \Gamma' \subseteq \Lambda_Q$ and such that Γ' is integral over R. This always exists, by Zorn's Lemma.

If ${}_R\Gamma$ is finitely generated, then Γ becomes a maximal R-order in Λ_Q. If Λ is commutative, then Γ is the normalization of Λ in the sense of commutative ring theory.

We let $\tilde{R}$ denote the integral closure of R in the center of Λ_Q. Then $R_Q \subseteq \tilde{R}_Q \subseteq \Lambda_Q$ and $\tilde{R}_Q$ is the total quotient ring of $\tilde{R}$.

Let X and Y be Λ-lattices (see Definitions 1.2) and Ω any ring such that $\Lambda \subseteq \Omega \subseteq \Lambda_Q$. Then

(2.4.1) $\quad X \cong Y$ (as Λ-modules) $\Rightarrow \Omega X \cong \Omega Y$ (as Ω-modules)

because any Λ-isomorphism $f: X \cong Y$ extends uniquely to a Λ_Q-isomorphism $X_Q \cong Y_Q$.

The next proposition shows that normalizations have the familiar ring-theoretic properties of maximal orders (e.g., 1-sided ideals are projective).

2.5. PROPOSITION (in the notation of 2.4). (i) *$\tilde{R}$ is a direct product of Dedekind domains.*

(ii) *$\tilde{R}$ is the center of Γ, and Γ is a maximal $\tilde{R}$-order in Λ_Q. Moreover, Γ is a direct product of maximal orders over Dedekind domains (the direct factors of $\tilde{R}$) in simple artinian rings.*

(iii) *Every maximal $\tilde{R}$-order containing in Λ_Q is a normalization of* 4.

Proof. Statement (i) is the Krull–Akizuki theorem [N, (33.2)]. Since the elements of $\tilde{R}$ commute with those of Γ, every element of $\tilde{R}\Gamma$ is integral over R. So maximality of Γ shows that $\tilde{R}\Gamma \subseteq \Gamma$, hence $\tilde{R} \subseteq \Gamma$. The rest of the proof is straightforward. ▮

2.6. PROPOSITION. *Let U, V be (finitely generated) Γ-modules. Then $U_{\mathfrak{n}} \cong V_{\mathfrak{n}}$ (as $\Gamma_{\mathfrak{n}}$-modules, $\forall$ maximal ideal $\mathfrak{n}$ of $\tilde{R}$) if and only if $U_{\mathfrak{m}} \cong V_{\mathfrak{m}}$ (as $\Gamma_{\mathfrak{m}}$-modules, $\forall$ maximal ideal $\mathfrak{m}$ of R).*

Note. Under the hypotheses of this proposition, U and V might not be finitely generated as modules over Λ or R. Nevertheless, this proposition allows us to speak unambiguously of the *genus* of U.

Proof. Since $\tilde{R}$ is integral over R, we can suppose that R is local. Then this is a special case of [EG, 2.3 and 2.5(ii)]. ▮

The next proposition gives the direct limit argument that we use to deal with the fact that Γ may not be a finitely generated R-module. Recall that $\mathrm{gen}(\Omega X)$ means the genus of the Ω-module ΩX.

2.7. PROPOSITION. (i) *Each finite subset F of Γ is contained in an R-overorder Ω of Λ ($\Lambda \subseteq \Omega \subseteq \Gamma$), that is, in a ring Ω such that ${}_R\Omega$ is finitely generated and $\Lambda \subseteq \Omega \subseteq \Gamma$.*

(ii) *Let X and Y be Λ-lattices such that $X_Q \cong Y_Q$ (as Λ_Q-modules). Then there is an R-overorder Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) such that $\Omega Y \in \mathrm{gen}(\Omega X)$ (as Ω-modules).*

(iii) *Let X be any Λ-lattice. Then ΩX is Ω-projective for all sufficiently large R-overorders Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) (i.e., for all overorders Ω containing some fixed overorder).*

Proof. (i) Let $\{\gamma_i\}$ be a finite set of generators of the $\tilde{R}$-module Γ, chosen so that $\{\gamma_i\}$ contains a finite set of generators of the R-module Λ. Let $\{x_k\}$ be the finite set of elements of $\tilde{R}$ that occur when (i) the elements of F are expressed in some way as $\tilde{R}$-linear combinations of the γ_i, and when (ii) all products $\gamma_i\gamma_j$ are expressed in some way as $\tilde{R}$-linear combinations of the γ_i. Since every x_k is integral over R, the ring $S = R[\{x_k\}]$ is a finitely generated R-module. Therefore the finitely generated R-module $\Omega = \Sigma_i S\gamma_i$ is a ring, and is the desired R-overorder of Λ.

(ii) Since $X_Q \cong Y_Q$, Lemma 2.1 yields

$$X_{\mathfrak{m}} \cong Y_{\mathfrak{m}} \qquad (\mathrm{a}\forall\mathfrak{m}). \tag{2.7.1}$$

Let E be the finite, exceptional set of $\mathfrak{m}$, at which isomorphism does not hold.

Choose an $\mathfrak{m} \in E$. We claim that there is an overorder Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) such that

$$(\Omega X)_{\mathfrak{m}} \cong (\Omega Y)_{\mathfrak{m}}. \tag{2.7.2}$$

Note that $\tilde{R}_Q$ is the total quotient ring of $\tilde{R}$, and all localizations of $\tilde{R}$ at its maximal ideals are discrete valuation rings. Since $X_Q \cong Y_Q$ we therefore have $(\Gamma X)_{\mathfrak{n}} \cong (\Gamma Y)_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of $\tilde{R}$. Hence by Proposition 2.6 we have $(\Gamma X)_{\mathfrak{m}} \cong (\Gamma Y)_{\mathfrak{m}}$ for the maximal ideal $\mathfrak{m}$ of R that we are working with. Choose one such $\Gamma_{\mathfrak{m}}$-isomorphism f: $(\Gamma X)_{\mathfrak{m}} \cong (\Gamma Y)_{\mathfrak{m}}$.

Let $\{x_i\}$ and $\{y_i\}$ be finite generating sets for the Λ-modules X and Y, respectively. Then there exist relations

$$(2.7.3) \qquad f(x_i/1) = (d/1)^{-1} \Sigma_j \gamma_{ij}(y_j/1)$$

and

$$y_j/1 = (d/1)^{-1} \Sigma_i \gamma'_{ij} f(x_i/1)$$

with $d \in R - \mathfrak{m}$ and each γ_{ij} and $\gamma'_{ij} \in \Gamma$. By part (i) of the present proposition, there is an R-overorder Ω of Λ containing d and every γ_{ij} and γ'_{ij}. By (2.7.3) the claim holds for Ω.

By (2.4.1) the isomorphism in (2.7.1) remains true if we multiply both X and Y by Ω, and both this and isomorphism (2.7.2) remain true when Ω is replaced by any larger R-order. Therefore, by repeated use of the claim, we can find a larger overorder Ω such that (2.7.2) holds for every $\mathfrak{m}$ in the finite set E, and the altered (2.7.1) (multiplied by Ω) also remains true.

(iii) Since Λ_Q is semisimple artinian, there is a Λ-lattice U such that $X_Q \oplus U_Q$ is a free Λ_Q-module; say $X_Q \oplus U_Q \cong (\Lambda^n)_Q$. By part (ii) of this proposition $\Omega(X \oplus U)$ is in the Ω-genus of Ω^n for some Ω and hence is Ω-projective. This remains true when Ω is replaced by any larger overorder, by (2.4.1). ∎

Our final result and its corollaries show that there is an overorder that behaves very much like a maximal order, with respect to any given finite number of Λ-lattices.

2.8. THEOREM. *Let Λ be an R-order in the semisimple artinian ring Λ_Q and write*

$$(2.8.1) \qquad \Lambda_Q \cong S_1^{c(1)} \times \cdots \times S_n^{c(n)}$$

with $S_1, ..., S_n$ pairwise nonisomorphic simple left Λ_Q-modules. Let $\mathcal{M}$ be a finite set of Λ-lattices. Then there is an R-overorder Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) with the following properties.

(i) *$\Omega = \Omega_1 \oplus \cdots \oplus \Omega_n$ (ring direct sum) where each Ω_i is an R-order in the simple artinian ring $(\Omega_i)_Q$ and the simple left $(\Omega_i)_Q$-module is S_i;*

(ii) *For each i, $\Omega_i = \bigoplus_j L_{ij}$ where each L_{ij} is a left ideal, such that $L_{ij} \in \mathrm{gen}(L_{i1})$ and $(L_{ij})_Q \cong S_i$ ($\forall j$); and*

(iii) *For each $M \in \mathcal{M}$ and each i, $\Omega_i M$ is Ω-projective and has a decomposition of the form $\Omega_i M = \bigoplus_{ih} M_{ih}$ with each $M_{ih} \in \mathrm{gen}(L_{i1})$.*

Proof. By (2.4.1) the desired properties of Ω remain true if Ω is replaced by any larger R-suborder in Λ_Q. Therefore we can assume that $\mathcal{M}$ consists of a single Λ-lattice M (and iterate the construction in the proof).

By Proposition 2.5, $\Gamma = \bigoplus_{i=1}^n \Gamma_i$ (ring direct sum) where each Γ_i is a maximal, central order over a Dedekind domain in the simple artinian ring

$(\Gamma_i)_Q$ and the simple left $(\Gamma_i)_Q$-module is S_i. Therefore [R, 27.4] there are elements $e_{ij} \in \Gamma_i$ such that

$$(2.8.2) \qquad \Gamma = \bigoplus_{ij} \Gamma e_{ij} \text{ with } e_{ij} \in \Gamma_i \qquad \text{and} \qquad (\Gamma e_{ij})_Q \cong S_i \ (\forall i, j).$$

By Proposition 2.7(i) there is an R-overorder Ω such that (2.8.2) holds with Γ replaced by Ω. Then, by Proposition 2.7(ii) we can enlarge Ω so that we also have $\Omega e_{ij} \in \text{gen}(\Omega e_{i1})$ $(\forall i, j)$. This proves (i) and (ii). It remains to show that the desired properties of $\Omega_i M$ can be obtained by further enlarging Ω.

Since Γ is a direct sum of maximal orders over Dedekind domains in simple artinian rings, we have a decomposition [R, 27.4]

$$(2.8.3) \qquad \Gamma M = \bigoplus_{ih} N_{ih} \qquad \text{with} \quad (N_{ih})_Q \cong S_i \ (\forall i)$$

for a suitable index set $\{h\}$. Since each N_{ij} is a finitely generated Γ-module, a suitable enlargement of Ω yields a relation of the form

$$(2.8.4) \qquad \Omega M = \bigoplus_{ih} P_{ih} \qquad \text{with} \quad (P_{ih})_Q \cong S_i \text{ and } P_{ih} \subseteq N_{ih} \ (\forall i).$$

Finally, repeated applications of Proposition 2.7 yield a further enlargement of Ω such that the Ω-module M_{ij} generated by P_{ij} belongs to the genus of Ωe_{i1} as desired. ∎

If each Ω_i really were a maximal order in a simple algebra, it would be the endomorphism ring of a projective module over an order in a division algebra. Our next result shows that Ω_i has this crucial property anyway.

2.9. Lemma. *Let Ω be an R-order in the simple artinian ring $A = \Omega_Q$, and suppose that $\Omega = \bigoplus_{j=1}^{n} L_j$ where each L_j is a left ideal in* gen(L_1) *and $(L_1)_Q$ is a minimal left ideal of A. Let $\Delta = \Omega$-*end(L_1) *(Ω-endomorphism ring). Then $\Omega \cong \Delta$-*end(L_1) *via $\omega \to \mu(\omega) =$ left multiplication by ω; and L_1 is a locally-free Δ-module of constant rank (at the maximal ideals of R).*

Proof. We can suppose that R is a local ring. We then have that each $L_j \cong L_1$ (as Ω-modules); so Ω can be identified with the $n \times n$ matrix ring over Δ-end(L_1). In this situation it is well known that μ is an isomorphism and L_1 is a free Δ-module of rank n. ∎

3. Units Action, Connection with $\mathbf{K}_1$

In this section Λ and Ω denote R-orders in the semisimple artinian ring Λ_Q such that $\Lambda \subseteq \Omega \subseteq \Lambda_Q$ $(= \Omega_Q)$.

3.1. DEFINITIONS [(Λ, Ω)-genus]. Let M be a Λ-lattice (see Definitions 1.2), so that $M \subseteq M_Q$. We define ΩM to be the Ω-submodule of M_Q generated by M. Note that ΩM is an Ω-lattice. Moreover, Ω-lattices (but not Ω-modules) M' and N' are isomorphic as Ω-lattices if and only they are isomorphic as Λ-lattices, because any Λ-isomorphism $M' \to N'$ can be uniquely extended to a Λ_Q-isomorphism $(M')_Q \to (N')_Q$.

We caution the reader that ΩM is not usually isomorphic to $\Omega \otimes_\Lambda M$. Therefore ΩM may not be the unique Ω-module generated by M. However, ΩM is the unique Ω-lattice generated by M, in the following sense. Any isomorphism $M \cong M'$ of Λ-lattices can be canonically extended to an isomorphism $\beta: \Omega M \cong \Omega M'$ [by first extending to an isomorphism $M_Q \cong (M')_Q$ and then restricting to ΩM].

By the (Λ, Ω)*-genus*, (Λ, Ω)-gen(M) of a Λ-lattice M we mean the collection of all Λ-lattices N in gen(M) such that $\Omega N \cong \Omega M$ (as Ω-modules or, equivalently, as Λ-modules, since M and N are lattices). This notion generalizes the *restricted genus* of M, which is defined when Λ is contained in a maximal R-order Γ in Λ_Q, and equals the (Λ, Γ)-genus of M.

Our objective in this section is to obtain a description of (Λ, Ω)-gen(M) in terms of the group of units of an artinian ring Ω/T, when ΩM is Ω-projective. We conclude by showing that the (Λ, Ω)-genus class group of M, that is, the group of stable isomorphism classes in (Λ, Ω)-gen(M), can be described by an exact sequence of $\mathbf{K}_1$-groups.

Let T denote a 2-sided ideal of Ω such that $T \subseteq \Lambda$ and such that Λ/T and Ω/T are R-modules of finite length, hence artinian rings. We call T a *conductor ideal* from Ω to Λ.

To see that such an ideal always exists, we can replace R by its image $R \cdot 1$ in Λ. Then R has no nilpotent elements $\neq 0$, because any such element would generate a nilpotent ideal of the semisimple artinian ring Λ_Q. Since Ω is contained in Λ_Q and Ω is a finitely generated R-module, there is a regular element d of R [see Subsection 1.1] such that $d\Omega \subseteq \Lambda$, so we can take $T = d\Omega$.

The ring Λ is the pullback of its *conductor square*, the first commutative square in (3.1.1), and M is the pullback of the second commutative square in (3.1.1) which, following Wiegand [W], we call the *standard pullback* for M (with respect to T).

$$(3.1.1) \qquad \begin{array}{ccc} \Lambda & \rightarrowtail & \Omega \\ \downarrow & & \downarrow \\ \Lambda/T & \rightarrowtail & \Omega/T \end{array} \qquad \begin{array}{ccc} M & \rightarrowtail & \Omega M \\ \downarrow & & \downarrow \\ M/TM & \rightarrowtail & \Omega M/TM \end{array}$$

In module diagrams, arrows with tails denote monomorphisms, and arrows with double heads indicate surjections. The monomorphisms in (3.1.1) denote inclusion.

The following lemma extends [W, 2.1] to our noncommutative setting.

3.2. LEMMA. *Suppose that the Λ-lattice M is the pullback of a commutative square of the form*

$$\begin{tikzcd} M\ (\Lambda\text{-lattice}) \arrow[r, tail] \arrow[d, twoheadrightarrow] & N\ (\Omega\text{-lattice}) \arrow[d, twoheadrightarrow] \\ Y\ (\Lambda/T\text{-module}) \arrow[r, tail] & N/TN \end{tikzcd} \tag{3.2.1}$$

where N/TN is generated, as an Ω-module by its Λ-submodule Y. Then (3.2.1) *is isomorphic to the standard pullback for M.*

Proof. Wiegand's proof (with our Ω replacing his $\tilde{R}$) works, provided we can define an isomorphism $\beta\colon \Omega M \cong N$. Our hypotheses on diagram (3.2.1) show that $N = \Omega M'$ where M' is the image of M in N. So the desired β arises from the uniqueness of ΩM, as explained in the second paragraph of Definitions 3.1. ∎

3.3. *Notation* (*Units Action, βM*). Let M be a Λ-lattice such that ΩM is Ω-projective. [If M is an arbitrary Λ-lattice, then ΩM is Ω-projective for all sufficiently large Ω, by Proposition 2.7.] For $\vartheta \in E^*(\Omega M/TM)$, the multiplicative group of units of the endomorphism ring $E(\Omega M/TM)$ of the Ω-module $\Omega M/TM$, define the Λ-module M^ϑ to be the pullback of the diagram

$$\begin{tikzcd} M^\vartheta \arrow[rr, tail] \arrow[d, twoheadrightarrow] & & \Omega M \arrow[d, twoheadrightarrow, "j"] \\ M/TM \arrow[r, tail, "i"] & \Omega M/TM \arrow[r, twoheadrightarrow, "\vartheta"] & \Omega M/TM \end{tikzcd} \tag{3.3.1}$$

where i and j are the maps in the standard pullback for M.

M^ϑ is a Λ-lattice because $(\Omega M)_Q = M_Q$.

Whenever we use the notation M^ϑ *it is understood that M is a Λ-lattice such that ΩM is Ω-projective, and $\vartheta \in E^*(\Omega M/TM)$.*

We let βM denote the inclusion i in (3.3.1), that is, the "bottom line" of the standard pullback (3.1.1) for M. Moreover, we let $E^*(\beta M)$ denote the set of $\vartheta \in E^*(\Omega M/TM)$ such that $\vartheta(M/TM) = M/TM$.

3.4. DOUBLE COSET THEOREM. *For any M, the correspondence $\vartheta \to M^\vartheta$ induces a one-to-one correspondence between the set of isomorphism classes of Λ-lattices M^ϑ in (Λ, Ω)-gen(M), and the set of double cosets*

$$E^*(\beta M)\backslash E^*(\Omega M/TM)/E^*(\Omega M), \tag{3.4.1}$$

where products $\vartheta\varphi$, with $\vartheta \in E^*(\Omega M/TM)$ and $\varphi \in E^*(\Omega M)$ are evaluated by first replacing φ by its natural image in $E^*(\Omega M/TM)$.

Proof. Let $(3.3.1)_\varphi$ denote the diagram obtained by replacing ϑ by φ in diagram (3.3.1).

Suppose $M^\vartheta \cong M^\varphi$. Then by Lemma 3.2 we have $(3.3.1)_\vartheta \cong (3.3.1)_\varphi$. Drawing one of these diagrams inside the other, and filling in the module isomorphisms that show the diagrams to be isomorphic, we see that $\varphi \in E^*(\beta M) \cdot \vartheta \cdot E^*(\Omega M)$ as desired. Conversely, the relation $\varphi \in E^*(\beta M) \cdot \vartheta \cdot E^*(\Omega M)$ shows that $(3.3.1)_\vartheta \cong (3.3.1)_\varphi$ and hence $M^\vartheta \cong M^\varphi$.

To see that $M^\vartheta \in \operatorname{gen}(M)$ we can suppose that R is a local ring, and therefore wish to show that $M^\vartheta \cong M^1$. It therefore suffices, by the previous paragraph, to show that we can lift the automorphism ϑ in (3.3.1) to an automorphism α of ΩM, as shown in

(3.4.2)
$$\begin{tikzcd}
\Omega M \arrow[r, tail, two heads, "\alpha"] \arrow[d, two heads, "j"] & \Omega M \arrow[d, two heads, "j"] \\
\Omega M/TM \arrow[r, tail, two heads, "\vartheta"] & \Omega M/TM
\end{tikzcd}$$

Let $E = E(\Omega M)$. Since 1 in in the stable range of the local ring R, it is also in the stable range of the module-finite R-algebra E. Since ΩM is Ω-projective, the natural map $\Phi\colon E \to E(\Omega M/TM)$ is a surjection, so $\vartheta = \Phi(\beta)$ for some $\beta \in E$. Let $K = \ker(\Phi)$. Then $E\beta + K = E$. Since 1 is in the stable range of E, there exists $\tau \in K$ such that $\alpha = \beta + \tau$ is a unit of E, and lifts ϑ, as desired.

Now return to the original, non-local R. It remains to show that every $N \in (\Lambda, \Omega)\text{-gen}(M)$ has the form M^ϑ for some ϑ. The diagram below shows the standard pullback for N inside of the square that defines M^ϑ for some ϑ not yet defined. We proceed to define ϑ and the arrows connecting the two squares, that define an isomorphism of pullback squares.

(3.4.3)
$$\begin{tikzcd}
M^\vartheta \arrow[rrrr, tail] \arrow[ddd, tail, two heads] \arrow[dr, "\varphi"] & & & & \Omega M \arrow[ddd, tail, two heads] \arrow[dl, "\beta"'] \\
 & N \arrow[rr, tail] \arrow[d, tail, two heads] & & \Omega N \arrow[d, tail, two heads] & \\
 & N/TN \arrow[rr, tail] & & \Omega N/TN & \\
M/TM \arrow[rr, tail, "i"] \arrow[ur, "\alpha"] & & \Omega M/TM \arrow[rr, tail, two heads, "\vartheta"] \arrow[ur, "\varepsilon"] & & \Omega M/TM \arrow[ul, "\gamma"']
\end{tikzcd}$$

Since $N \in \text{gen}(M)$ the bottom lines βM and βN are isomorphic. Let α and ε be isomorphisms that, together with i and the fourth, unlabeled map, form a commutative square. By hypothesis $\Omega M \cong \Omega N$. Let β be any such isomorphism, and let γ be the isomorphism induced by β. Then let ϑ be the isomorphism that makes the triangle containing ϑ, ε, γ commute. This defines M^{ϑ}, the pullback of the outer square.

If we ignore M^{ϑ} and N, all triangles and squares now commute. Therefore we have an isomorphism of the pullbacks determined by the inner and outer squares. In particular, there is an isomorphism $\varphi\colon M^{\vartheta} \cong N$, as desired. ∎

We call an automorphism ε of a module U *elementary* with respect to a decomposition $U = V \oplus W$ if $\varepsilon = 1 + \sigma$ where $\sigma(V) \subseteq W$ and $\sigma(W) = 0$, or $\sigma(W) \subseteq V$ and $\sigma(V) = 0$.

3.5. COROLLARY. *Let $M = X \oplus Y$, and let $\varepsilon \in E^*(\Omega M/TM)$ be elementary with respect to the decomposition $\Omega M/TM = \Omega X/TX \oplus \Omega Y/TY$. Then*

(i) *ε belongs to the image of $E^*(\Omega M)$ in $E^*(\Omega M/TM)$; and*

(ii) *$M^{\vartheta\varepsilon} \cong M^{\vartheta}$ for all ϑ.*

Proof. Let $\varepsilon = 1 + \sigma$ where $\sigma(\Omega X/TX) \subseteq \Omega Y/TY$. Since ΩM is Ω-projective, so are ΩX and ΩY. Hence σ is the natural image of some $\tau \in \text{hom}(X, Y)$. Then $\beta = 1 + \tau \in E^*(\Omega M)$ and β lifts ε. This proves (i). Statement (ii) now follows from the Double Coset Theorem. ∎

3.6. COROLLARY. *Let*

$$(3.6.1) \qquad M = \bigoplus_{i=1}^{n} X_i, \qquad \text{where} \quad X_i/TX_i \cong X_1/TX_1 \ (\forall i).$$

Then every isomorphism class in (Λ, Ω)-gen(M) is represented by a lattice of the form $\bigoplus_{i=1}^{n} (X_i)^{\sigma_i}$ with $\sigma_i \in E^*(\Omega X_i/TX_i)$. If we consider the isomorphisms in (3.6.1) *to be equality, we have*

$$(3.6.2) \qquad \bigoplus_{i=1}^{n} (X_i)^{\sigma_i} \cong (X_1)^{\delta} \oplus X_2 \oplus \cdots \oplus X_n, \qquad \text{where} \quad \delta = \sigma_1 \sigma_2 \cdots \sigma_n.$$

Proof. Every module in (Λ, Ω)-gen(M) is isomorphic to M^{ϑ} for some ϑ, by the Double Coset Theorem. By hypothesis, we can identify $E(\Omega M/TM)$ with the ring of $n \times n$ matrices over the ring $E = E(\Omega X_1/TX_1)$. Since E is an artinian ring, it has 1 in its stable range, so $\vartheta = \sigma\varepsilon$ where σ is a diagonal matrix and ε is a product of elementary matrices. By Corollary 3.5, $M^{\sigma\varepsilon} \cong M^{\sigma}$, and this proves the first assertion of the present corollary.

The second assertion results from the fact that $\sigma = \mathrm{diag}(\delta, 1, 1, ..., 1) \cdot \alpha$ where, as above, α is a product of elementary matrices. ▮

3.7. LEMMA. *Let $\Lambda \subseteq \Omega \subseteq \Gamma$, where Λ and Ω are R-orders in the semisimple artinian ring Λ_Q and Γ is a normalization of Λ. Let M be a Λ-lattice such that ΩM is Ω-projective. Then*

(i) *Every Ω-lattice in* $\mathrm{gen}(\Omega M)$ *is isomorphic to ΩN for some* $N \in \mathrm{gen}(M)$.

(ii) *Every Γ-lattice in* $\mathrm{gen}(\Gamma M)$ *is isomorphic to ΓN for some* $N \in \mathrm{gen}(M)$.

(iii) $M_Q \cong N_Q \Leftrightarrow \Gamma N \in \mathrm{gen}(\Gamma M)$ *for any Λ-lattice N.*

(*See Proposition* 2.6 *to clarify the terminology here.*)

Proof. We can suppose that $R \subseteq \Lambda$, hence R has no nilpotent elements $\neq 0$. Consequently, the set of regular elements of R is $R - \bigcup \min \mathrm{spec}(R)$, the complement of union the set of minimal prime ideals of R.

(i) This proof makes use of the *regular localizations* $M_{\mathfrak{m}\rho}$ at maximal ideals $\mathfrak{m}$ of R, where

(3.7.1) $$M_{\mathfrak{m}\rho} = \{d^{-1}m \mid m \in M \text{ and } d \text{ is a regular element of } R - \mathfrak{m}\}.$$

Since only regular elements occur as denominators, and M is a lattice, the canonical maps $M \to M_{\mathfrak{m}\rho} \to M_Q$ are monomorphisms, and we regard them as inclusions. (See [GL, 2.3] for more about this.)

We claim, for Λ-lattices M and N, that $N \in \mathrm{gen}(M)$ if and only if $M_{\mathfrak{m}\rho} \cong N_{\mathfrak{m}\rho}$ for all maximal ideals of R. The "if" part holds since $(M_{\mathfrak{m}\rho})_{\mathfrak{m}} \cong M_{\mathfrak{m}}$. The converse follows from the canonical isomorphism

(3.7.2) $$R_{\mathfrak{m}\rho} \cong R_{\mathfrak{m}} \oplus \{R_{\mathfrak{p}} \mid \mathfrak{p} \in \min \mathrm{spec}(R) \text{ and } \mathfrak{p} \not\subseteq \mathfrak{m}\}.$$

This, in turn, holds since every such $\mathfrak{p}$ becomes a maximal ideal, as well as a minimal prime, in $R_{\mathfrak{m}\rho}$; so the field $R_{\mathfrak{p}}$ is a direct factor of $R_{\mathfrak{m}\rho}$.

Let $V = M_Q$. We define a *full Λ-lattice in V* to be a Λ-lattice $N \subseteq V$ such that $N_Q = V$. The Strong Consistency Theorem [GL, 2.6] states the following. Let $\mathcal{M}$ be a finite set of maximal ideals of R, and for each $\mathfrak{m} \in \mathcal{M}$ let $N(\mathfrak{m})$ be a full $\Lambda_{\mathfrak{m}\rho}$-lattice in V. Then there is a unique full Λ-lattice N in V such that $N_{\mathfrak{m}\rho} = N(\mathfrak{m})$ for $\mathfrak{m} \in \mathcal{M}$ and $N_{\mathfrak{m}\rho} = M_{\mathfrak{m}\rho}$ otherwise.

Now take any $U \in \mathrm{gen}(\Omega M)$. After replacing U by its image in V we have that U is a full Ω-lattice in V. Choose a regular element $d \in R$ such that $d\Omega \subseteq \Lambda$. Then R/Rd is an artinian ring, so there are only finitely many maximal ideals $\mathfrak{m}$ of R that contain d. For each such $\mathfrak{m}$, let $N(\mathfrak{m})$ be the image of $M_{\mathfrak{m}\rho}$ under some isomorphism $(\Omega M)_{\mathfrak{m}\rho} \cong U_{\mathfrak{m}\rho}$. Then $N(\mathfrak{m})$ is a full $\Lambda_{\mathfrak{m}\rho}$-lattice in V and $\Omega_{\mathfrak{m}\rho}N(\mathfrak{m}) = U_{\mathfrak{m}\rho}$. By the Strong Consistency Theorem there is a Λ-lattice N such that $N_{\mathfrak{m}\rho} = N(\mathfrak{m})$ whenever $d \in \mathfrak{m}$ and such that $N_{\mathfrak{m}\rho} = U_{\mathfrak{m}\rho}$ otherwise. Since $\Omega_{\mathfrak{m}\rho} = \Lambda_{\mathfrak{m}\rho}$ when $d \notin \mathfrak{m}$, N is the desired Λ-lattice.

(ii) Let W be a Γ-lattice in $\mathrm{gen}(\Gamma M)$, and let W' be the Λ-lattice generated by some finite set of Γ-generators of W; so $W = \Gamma W'$. Then $(W')_Q = W_Q \cong M_Q$. So, by Proposition 2.7, there is an R-order Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) such that $\Omega W' \in \mathrm{gen}(\Omega M)$. By part (i) of the present lemma we therefore have $\Omega W' \cong \Omega N$ for some $N \in \mathrm{gen}(\Omega M)$. We now have $\Gamma N \cong \Gamma W' = W$, as desired.

(iii) We can suppose that R is the center of Λ, by Subsection 1.3; and by Proposition 2.6 we can suppose that $R = \tilde{R}$, so R is a direct product of Dedekind domains. Since we can also suppose that R is local, R is now a discrete valuation ring. Thus Γ is a central, maximal order over a discrete valuation ring, a situation in which the result we are proving is well known [R, 18.7(i)]. ∎

3.8. *Remark* (*Connection with* $\mathbf{K}_1$). Let M be a Λ-lattice such that ΩM is Ω-projective. We say that a Λ-lattice $N \in \mathrm{gen}(M)$ is *stably isomorphic* to M if $N \oplus M \cong M \oplus M$. The stable isomorphism classes $[N]$ of modules $N \in \mathrm{gen}(M)$ can be made into an abelian group $\mathscr{G}(M)$, the *genus class group* of M, by defining $[S] = [N] + [N']$ to mean that $M \oplus S \cong N \oplus N'$. See [GL, 1.4].

Let $(\Lambda, \Omega)\text{-}\mathscr{G}(M) = \{[N] \in \mathscr{G}(M) \mid [\Omega N] = [\Omega M]\}$, where $[\Omega N] = [\Omega M]$ means that $\Omega N \oplus \Omega N \cong \Omega M \oplus \Omega M$. This is a subgroup of $\mathscr{G}(M)$, which we call the *(Λ, Ω)-genus class group* of M. The connection of our units action with $\mathbf{K}_1$ is given by:

3.9. THEOREM. *For M as above (and other notation as in* 3.1), *there is an exact sequence of groups*

$$(3.9.1) \qquad \mathbf{K}_1(\beta M) \times \mathbf{K}_1(\Omega M) \xrightarrow{\ \nu\ } \mathbf{K}_1(E(\Omega M/TM)) \xrightarrow{\ \sigma\ } (\Lambda, \Omega)\text{-}\mathscr{G}(M).$$

Proof. To define ν, note that elements $\varphi \in GL_n(E(\beta M))$ and $\vartheta \in GL_n(E(\Omega M))$ have natural images in $GL_n(E(\Omega M/TM))$. Then define ν to be the map that sends each ordered pair $([\varphi], [\vartheta])$ to the product $(\nu\varphi)(\nu\vartheta)$ of natural images in $\mathbf{K}_1(...)$. To define σ let $\gamma \in GL_n(E(\Omega M/TM))$, for some n. Since 1 in in the stable range of the artinian ring $E(\Omega M/TM)$, we have $[\gamma] = [\delta]$ [images in $\mathbf{K}_1(...)$] for some $\delta \in E^*(\Omega M/TM)$. Set $\sigma[\gamma] = [M^\delta]$.

To see that σ is well-defined, recall that since 1 is in the stable range of $E(\Omega M/TM)$ the kernel of the natural map $GL_2(E(\Omega M/TM)) \to \mathbf{K}_1(E(\Omega M/TM))$ is generated by elementary matrices. Therefore, if $[\gamma]$ also equals $[\delta']$ we have $\mathrm{diag}(\delta, 1) = \mathrm{diag}(\delta', 1) \cdot \varepsilon$ where ε is a product of elementary matrices. But then $M^{\delta} \oplus M \cong M^{\delta'} \oplus M$ by Corollary 3.5.

A similar argument, using Corollary 3.6 shows that σ is a group homomorphism. To see that σ is a surjection, let $[N] \in (\Lambda, \Omega)\text{-}\mathscr{G}(M)$. Then $\Omega(N \oplus M) \cong \Omega(M \oplus M)$ so $N \oplus M \in (\Lambda, \Omega)\text{-gen}(M \oplus M)$. Therefore $N \oplus M \cong (M \oplus M)^{\mathrm{diag}(\alpha, 1)} \cong M^{\alpha} \oplus M$ for some α, by Corollary 3.6. So $[N] = [M^{\alpha}] \in (\Lambda, \Omega)\text{-}\mathscr{G}(M)$.

Moreover, $\ker(\sigma) = \{[\delta] \in \mathbf{K}_1(E(\Omega M/TM)) \mid \delta \in E^*(\Omega M/TM)$ and $M^{\delta} \oplus M \cong M \oplus M\}$. By our Double Coset Theorem, applied to $M \oplus M$ in place of M, $\ker(\sigma)$ is therefore the set of δ such that $\mathrm{diag}(\delta, 1) = \varphi\vartheta$ with $\varphi \in GL_2(E(\beta M))$ and $\vartheta \in GL_2(E(\Omega M))$. To see that $\mathrm{im}(\nu) = \ker(\sigma)$, as claimed, it therefore suffices to observe that the images of $GL_2(E(\beta M))$ and $GL_2(E(\Omega M))$ in $\mathbf{K}_1(E(\Omega M/TM))$ equal the respective images of $\mathbf{K}_1(E(\beta M))$ and $\mathbf{K}_1(E(\Omega M))$ there. These hold because $E(\beta M)$ and $E(\Omega M)$ are module-finite algebras over the ring R of Krull dimension 1. ∎

4. Local Versus Global Direct Summands

The following theorem is well known for lattices over the orders that occur in integral representation theory [CR1, (31.12)].

The letter $\mathfrak{m}$ always denotes a maximal ideal of R, and the subscript $\mathfrak{m}\rho$ denotes "regular localization at $\mathfrak{m}$." For an explanation of this, see the proof of Lemma 3.7.

4.1. Theorem. *Let X and M be Λ-modules such that $X_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ for all $\mathfrak{m}$. Then*

(i) *$X \mid M'$ for some $M' \in \mathrm{gen}(M)$; and*

(ii) *$X' \mid M$ for some $X' \in \mathrm{gen}(X)$.*

Proof. By Lemmas 1.4 and 1.6 we can assume that Λ is an R-order in the semisimple artinian ring Λ_Q and M and X are Λ-lattices. We can also suppose that R is the center of Λ.

(i) Since $X_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ we also have $X_{\mathfrak{m}\rho} \mid M_{\mathfrak{m}\rho}$ by [GL, (2.3.2)]. So there is a $\Lambda_{\mathfrak{m}\rho}$-lattice $Y(\mathfrak{m})$ such that

$$X_{\mathfrak{m}\rho} \oplus Y(\mathfrak{m}) \cong M_{\mathfrak{m}\rho}. \tag{4.1.1}$$

It suffices to show that there is a Λ-module Y such that $Y_{\mathfrak{m}\rho} \cong Y(\mathfrak{m})$ ($\forall \mathfrak{m}$), for then $X \oplus Y \in \mathrm{gen}(M)$ as desired. We do this by means of the Strong Consistency Theorem. (See the proof of Lemma 3.7.)

We have $(X_{\mathfrak{m}\rho})_Q \cong X_Q$ and $(M_{\mathfrak{m}\rho})_Q \cong M_Q$. Since the ring $A = \Lambda_Q$ is semisimple artinian, localizing (4.3.1) at Q shows that the A-isomorphism class of $Y(\mathfrak{m})_Q$ is independent of $\mathfrak{m}$. Choose an A-module V isomorphic to these modules $Y(\mathfrak{m})_Q$. Let Y' be any Λ-lattice such that $(Y')_Q = V$. Then $X_Q \oplus (Y')_Q \cong M_Q$. By Lemma 2.1 we have $X_{\mathfrak{m}} \oplus (Y')_{\mathfrak{m}} \cong M_{\mathfrak{m}}$ hence $X_{\mathfrak{m}\rho} \oplus (Y')_{\mathfrak{m}\rho} \cong M_{\mathfrak{m}\rho}$ (a$\forall\mathfrak{m}$). Since direct-sum cancellation of modules holds for module-finite algebras over semilocal rings [GW, EG], comparing this with (4.3.1) shows that $(Y')_{\mathfrak{m}\rho} \cong Y(\mathfrak{m})$ (a$\forall\mathfrak{m}$).

The Strong Consistency Theorem now shows that we can change the remaining finite number of localizations of Y' to $Y(\mathfrak{m})$, thus completing the proof of (i).

(ii) Let Γ be a normalization of Λ. Then, by Proposition 2.5, Γ is a maximal $\tilde{R}$-order, $\tilde{R}$ is a direct product of Dedekind domains, and hence Γ is a direct product of maximal orders over Dedekind domains. Moreover, these maximal $\tilde{R}$-orders are central, since R is the center of Λ. The theorem we are proving is well known in this situation (e.g., combine [R, (21.5) and (18.7)]). Therefore there is a Γ-lattice Y such that

(4.1.2) $$Y \mid \Gamma M \quad \text{and} \quad Y \in \operatorname{gen}(\Gamma X).$$

Here we are using the fact that $\operatorname{gen}(\Gamma X)$, computed with respect to maximal ideals of R is the same as $\operatorname{gen}(\Gamma X)$, computed with respect to maximal ideals of $\tilde{R}$, by Proposition 2.6.

Next we show that there is a Λ-lattice X_1 and an R-overorder Ω ($\Lambda \subseteq \Omega \subseteq \Gamma$) such that

(4.1.3) $$\Omega X_1 \mid \Omega M \quad \text{and} \quad X_1 \in \operatorname{gen}(X).$$

We have $Y = \Gamma X_1$ for some $X_1 \in \operatorname{gen}(X)$ by Lemma 3.7. Writing our relation $\Gamma X_1 \mid \Gamma M$ in terms of generators of X_1 and M involves only a finite number of elements of Γ. Therefore the existence of the desired order Ω follows from Proposition 2.7, and we can choose Ω such that ΩM is Ω-projective.

After replacing X by X_1 we get a decomposition $\Omega M = \Omega X \oplus V$ for some Ω-lattice V.

Let $M' \cong X \oplus Y'$ be a decomposition afforded by assertion (i) of the theorem, which we have already proved. Then $\Omega M' \cong \Omega X \oplus \Omega Y'$. So local cancellation shows that $V \in \operatorname{gen}(\Omega Y')$. We have $V = \Omega Y$ for some $Y \in \operatorname{gen}(Y')$, by Lemma 3.7.

Now we have $\Omega M \cong \Omega(X \oplus Y)$ so $M \in (\Lambda, \Omega)\text{-gen}(X \oplus Y)$. Let T be a conductor ideal from Ω to Λ, as defined in 3.1. Then, by the Double Coset Theorem 3.4, we have $M \cong (X \oplus Y)^{\vartheta}$ for some $\vartheta \in E^*((X \oplus Y)/T(X \oplus Y))$.

We can view ϑ as right multiplication by a 2×2 "matrix" ϑ whose entries are homomorphisms between the coordinate modules of $X \oplus Y$. Since the artinian rings $E(X/TX)$ and $E(Y/TY)$ have 1 in their stable range, there is a factorization $\vartheta = \mathrm{diag}(\delta, \gamma) \cdot \varepsilon$ where ε is a product of elementary matrices. This is well known when ϑ is an actual 2×2 matrix; and the details in this more general situation are worked out in the proof of [G2, 3.4].

We now have $M \cong (X \oplus Y)^{\vartheta} \cong (X \oplus Y)^{\mathrm{diag}(\delta, \gamma)}$ by Corollary 3.5. Since this is isomorphic to $X^{\delta} \oplus Y^{\gamma}$ our proof is complete. ∎

As immediate consequences of this theorem we have our generalization of Jacobinski's theorems [J] that indecomposability and direct-sum decompositions are genus properties. (The induction argument needed in 4.3 uses the fact that direct-sum cancellation holds for Λ-modules when R is local. See, e.g., [EG, 2.5] or [GW].)

4.2. COROLLARY. *Let M and N be Λ-modules in the same genus. Then M is indecomposable if and only if N is indecomposable.*

4.3. COROLLARY. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a decomposition of a Λ-module, and let $N \in \mathrm{gen}(M)$. Then there is a decomposition $N = \bigoplus_{i=1}^{n} N_i$ with each $N_i \in \mathrm{gen}(M_i)$.*

5. CANCELLATION IN A GENUS

5.1. LEMMA. *Let Δ be a module-finite R-algebra, and $E = E(M_{\Delta})$ where M is a locally free right Δ-module of rank $m \geqslant 2$. Then the natural map from E^* to $\mathbf{K}_1(E)$ is a surjection.*

Proof. We write elements of E as left operators on M.

Let N be a right Δ-module of the form $N = \bigoplus_i N_i$ with each $N_i \in \mathrm{gen}(\Delta)$, and let $\mathscr{E}(N)$ be the subgroup of $E^*(N)$ generated by all automorphisms φ of M such that φ is *elementary* with respect to some decomposition of this form [i.e., $N = \bigoplus_i H_i$ with each $H_i \in \mathrm{gen}(\Delta)$, and $\varphi = 1 + \alpha$ where $\alpha \in \hom(H_i, H_j)$ for some $i \neq j$].

The following is proved in [GL, 2.12].

(5.1.1) Let N be as above. Suppose that $L \cong L'$ for locally free direct summands L and L' of N, of constant rank (at the maximal ideals of R), such that $\mathrm{rank}(N) - \mathrm{rank}(L) \geqslant 2$. Then $L' = \alpha L$ for some $\alpha \in \mathscr{E}(N)$.

Since E is a module-finite algebra over the noetherian ring R of Krull dimension 1, every element of $\mathbf{K}_1(E)$ is the natural image $\nu(\vartheta)$ of some element $\vartheta \in GL_2(E) = E^*((M_{\Delta})^2)$ [B, p. 240].

Consider the Δ-module decomposition $M^2 = e_{11}M^2 \oplus e_{22}M^2$ where the

e_{ii} are matrix units. By (5.1.1), with $N = M^2$, we have $\vartheta(e_{11}M) = \alpha(e_{11}M)$ for some $\alpha \in \mathscr{E}(M^2)$. Therefore

$$\alpha^{-1}\vartheta = \begin{bmatrix} \gamma_1 & \gamma_2 \\ 0 & \gamma_3 \end{bmatrix} \in GL_2(E).$$

We therefore have $\nu(\vartheta) = \nu(\gamma_1)\,\nu(\gamma_3)\,\nu(\alpha)$. So it suffices to show that $\nu(\alpha) = 1$. It suffices to check this when α is an elementary automorphism with respect to some decomposition $M^2 = \bigoplus_i N_i$ with each $N_i \in \text{gen}(\Delta)$, and this follows easily from the abstract definition of $\mathbf{K}_1$ in terms of ordered pairs (P, φ) where P is a projective module and $\varphi \in E^*(P)$. See, e.g., [CR2, Definition 38.28]. ▮

5.2. Lemma. *Cancellation holds in* gen(M) *if* $L \oplus N \cong N \oplus N$ *implies* $L \cong N$ *whenever* L *and* N *are in* gen(M).

Proof. The following slight variation of Bass's cancellation theorem in Krull dimension 1 is proved in [GL, 1.3]. Suppose that $N, H, H_1, ..., H_n$ are Λ-modules in gen(M). Then

(5.2.1) $\quad M \oplus H_1 \oplus \cdots \oplus H_n \cong N \oplus H_1 \oplus \cdots \oplus H_n \Rightarrow M \oplus H \cong N \oplus H.$

Now suppose that $L \oplus X \cong N \oplus X$ with L, N, X in gen(M). We can replace X by N in this isomorphism, by (5.2.1), then use the hypothesis to cancel N. ▮

5.3. *Notation.* Let Λ be an R-order in the semisimple artinian ring Λ_Q, and M a left Λ-lattice. Let Ω be an R-overorder $(\Lambda \subseteq \Omega \subseteq \Lambda_Q)$ such that ΩM is Ω-projective, and let T be a conductor ideal from Ω to Λ, as defined in Notation 3.1.

The proofs in this section will make considerable use of the following notation, mostly from Section 3, which is most easily remembered by relating it to the second pullback diagram in (3.1.1).

$E(\Omega M/TM)$ denotes the endomorphism ring of the Ω-module $\Omega M/TM$.

$\beta(M)$ denotes the "bottom line" of M, that is, the inclusion $M/TM \subseteq \Omega M/TM$ in (3.1.1); and $E(\beta M)$ denotes the endomorphism ring of βM, that is, the set of all $\vartheta \in E(\Omega M/TM)$ such that $\vartheta(M/TM) \subseteq M/TM$.

$\lambda(M) := E^*(\beta M) \cdot E^*(\Omega M)$, where * denotes "units of" and these products $\vartheta\varphi$ are evaluated by first replacing φ by its natural image in $E(\Omega M/TM)$. Note that we do not claim that this set of products is a group.

Finally, ν denotes the natural homomorphism of any (specified) group into some specified Whitehead group $\mathbf{K}_1(...)$.

Let $N \in \text{gen}(M)$. Since the R-module Ω/T has finite length, we have $\Omega N/TN \cong \Omega M/TM$, hence $\mathbf{K}_1(E(\Omega N/TN)) \cong \mathbf{K}_1(E(\Omega M/TM))$.

We need to compare the conditions for cancellation in $\operatorname{gen}(M)$ and in $\operatorname{gen}(\Omega M)$. A result of Fröhlich [S1, p. 167] states that, for the orders that occur in integral representation theory, cancellation in $\operatorname{gen}(M)$ implies cancellation in $\operatorname{gen}(\Omega M)$. The following result extends this.

5.4. COMPARISON LEMMA. *Let Λ, M, Ω be as in Notation 5.3. Cancellation holds in* $\operatorname{gen}(M)$ *if and only if the following conditions hold for every* $N \in \operatorname{gen}(M)$.

(i) *Cancellation holds in* $\operatorname{gen}(\Omega M)$;

(ii) *For $\vartheta \in E^*(\Omega N/TN)$, $\nu(\vartheta) = 1$ in $\mathbf{K}_1(E(\Omega N/TN))$ implies $\vartheta \in \lambda(N)$;*

and

(iii) $\nu(GL_2(E(\Omega N))) \subseteq \nu\lambda(N)$ *in* $\mathbf{K}_1(E(\Omega N/TN))$.

Moreover, when the conditions hold, $\lambda(N)$ is a group, in fact a normal subgroup of $E^*(\Omega N/TN)$.

Proof. Suppose that cancellation holds in $\operatorname{gen}(M)$. To obtain condition (i), first recall from Lemma 3.7 that every Ω-lattice in $\operatorname{gen}(\Omega M)$ is isomorphic to ΩN for some $N \in \operatorname{gen}(M)$. So the hypothesis, here, can be stated $\Omega L \oplus \Omega X \cong \Omega N \oplus \Omega X$ with L, N, X in $\operatorname{gen}(M)$. This shows that $L \oplus X \in (\Lambda, \Omega)\text{-genus}(N \oplus X)$, in the notation of 3.1. So, by Corollary 3.6 we have

$$L \oplus X \cong N^\delta \oplus X \qquad \text{for some} \quad \delta \in E^*(\Omega N/TN).$$

Since cancellation holds in $\operatorname{gen}(M)$ we have $L \cong N^\delta$, and therefore $\Omega L \cong \Omega N^\delta \cong \Omega N$, where the last isomorphism holds since, by Theorem 3.4, $N^\delta \in (\Lambda, \Omega)\text{-gen}(N)$.

(ii) Since $\nu(\vartheta) = 1$, the diagonal matrix $\operatorname{diag}(\vartheta, 1) \in GL_2(E(\Omega N/TN))$ satisfies $\nu(\operatorname{diag}(\vartheta, 1)) = 1$. Since the artinian ring $E(\Omega M/TM)$ has 1 in its stable range, $\operatorname{diag}(\vartheta, 1)$ is a product of elementary matrices. So $(N \oplus N)^{\operatorname{diag}(\vartheta, 1)} \cong N \oplus N$ by Corollary 3.5. Then $N^\vartheta \oplus N \cong N \oplus N$, so cancellation in $\operatorname{gen}(N)$ yields $N^\vartheta \cong N$ and therefore $\vartheta \in \lambda(N)$ by Theorem 3.4.

(iii) Take $\vartheta \in GL_2(E(\Omega N))$ and let ϑ' be the natural image of ϑ in $GL_2(\Omega N/TN)$. Then $(N \oplus N)^{\vartheta'} \cong N \oplus N$ by Theorem 3.4 applied to $N \oplus N$ in place of M.

Since the artinian ring $E(\Omega N/TN)$ has 1 in its stable range, we have $\vartheta' = \operatorname{diag}(\delta, 1) \cdot \varepsilon$ where ε is a product of elementary matrices. Therefore, by Corollaries 3.5 and 3.6 we have $(N \oplus N)^{\vartheta'} \cong N^\delta \oplus N$. By the previous paragraph and cancellation in $\operatorname{gen}(M)$ we get $N^\delta \cong N$, which is equivalent to $\delta \in \lambda(N)$. Therefore $\nu(\vartheta) = \nu(\delta) \in \nu\lambda(N)$, as desired.

Conversely, we now assume conditions (i)–(iii) and establish cancellation in gen(M). By Lemma 5.2 it suffices to assume

$$(5.4.1) \qquad L\oplus N\cong N\oplus N \qquad \text{with } L \text{ and } N \text{ in gen}(M)$$

and then prove that $L\cong N$. Multiplying by Ω and using condition (i) shows that $\Omega L\cong\Omega N$. So $L\in(\Lambda,\Omega)$-gen(N), yielding that $L\cong N^{\delta}$ for some $\delta\in E^*(\Omega N/TN)$.

The isomorphism in (5.4.1) therefore implies that diag$(\delta, 1)\in\lambda(N\oplus N)$, and therefore

$$(5.4.2) \qquad \begin{aligned} v(\delta) &= v(\mathrm{diag}(\delta, 1)\in v\lambda(N\oplus N) \\ &= v(GL_2(E(\beta(N))))\cdot vGL_2(E(\Omega N)). \end{aligned}$$

Next we claim that

$$(5.4.3) \qquad v(\delta)\in v\lambda(N).$$

Since $\mathbf{K}_1$ is an abelian group, it suffices to show that each of the two factors at the extreme right-hand side of (5.4.2) is contained in $v\lambda(N)$. The ΩN-factor has this property by condition (iii). Consider the β-factor. Since the artinian ring $E(\beta(N\oplus N))$ has 1 in its stable range, any element φ of $GL_2(E(\beta(N)))$ has the form $\varphi=\mathrm{diag}(\alpha, 1)\cdot\vartheta$ where ϑ is a product of elementary matrices, and therefore $v(\vartheta)=1$. Therefore $v(\mathrm{diag}(\alpha, 1))=v(\alpha)\in vE^*(\beta N)\subseteq v\lambda(N)$ and (5.4.3) is proved.

Next we claim that $\delta\in\lambda(N)$. By (5.4.3) we have $\delta\in E^*(\beta N)\cdot E^*(\Omega N)\cdot$ (ker v); and by normality of ker(v) this equals $E^*(\beta N)\cdot(\ker v)\cdot E^*(\Omega N)$ which, by (ii), is contained in $\lambda(N)$. Thus the claim is proved, and we have $L\cong N$, establishing cancellation in gen(M).

Finally we prove the supplementary statement; so we can assume that cancellation holds in gen(M). Therefore the elements of the genus class group $\mathscr{G}(N)$ are actual (rather than stable) isomorphism classes of Λ-modules. (See Remark 3.8). The function $c\colon E^*(\Omega N/TN)\to\mathscr{G}(N)$ defined by $c(\vartheta)=[N^{\vartheta}]$ is a homomorphism by Corollary 3.6, and its kernel is $\lambda(N)$ by the Double Coset Theorem 3.4. So $\lambda(N)$ is a normal subgroup. ∎

5.5. Lemma. *Let $A=D\times D$ where D is a division R_Q-algebra, and let $\Omega=\Delta\times\Delta$ where Δ is an R-order in D. Then every R-suborder Λ of Ω satisfies cancellation in* gen(Λ) *if and only if the following conditions are satisfied for every nonzero ideal S of Δ.*

(i) *Δ satisfies cancellation in* gen(Δ);

(ii) *For* $\vartheta \in (\Delta/S)^*$, $v(\vartheta) = 1$ *in* $\mathbf{K}_1(\Delta/S)$ *implies that* ϑ *is in the image of* Δ^* *in* Δ/S; *and*

(iii) $v(GL_2(\Delta)) = v(\Delta^*)$ *in* $\mathbf{K}_1(\Delta/S)$.

Proof. Suppose that every suborder satisfies cancellation in its genus, and let S be given. Identify R with its image $R \cdot 1$ in Ω, and let $\Lambda = R + (S \times S)$, an R-order in $D \times D$. We are going to apply the Comparison Lemma, with conductor ideal $T = S \times S$ and $M = \Lambda$. In the notation that precedes that lemma, note that we then have $E^*(\beta M) = (\Lambda/T)^*$ and therefore

$$\lambda(M) = [\text{image of } R \text{ in } (\Delta/S) \times (\Delta/S)]^* \cdot (\Delta \times \Delta)^*. \tag{5.5.1}$$

Condition (ii). Take $\vartheta \in (\Delta/S)^*$ with $v(\vartheta) = 1$. We have $(\vartheta, 1) \in (\Omega/(S \times S))^*$. By the Comparison Lemma and (5.5.1) we get a factorization $(\vartheta, 1) = (r + S, r + S) \cdot (\alpha, \beta)$ with $r \in R$ and $\alpha, \beta \in \Delta^*$. Therefore $r \in \operatorname{im}(\Delta^*)$, hence $\vartheta \in \operatorname{im}(\Delta^*)$ as claimed.

Condition (iii). Take $\vartheta \in GL_2(\Delta)$. By the Comparison Lemma we get a factorization $v(\vartheta, I_2) = v(r + S, r + S) \cdot v(\alpha, \beta)$ and then finish the proof as in (ii).

Since condition (i) is obvious, the "only if" part of the proof is complete.

Conversely, suppose conditions (i)–(iii) hold, and let Λ be an R-suborder of Ω. Let T be a conductor ideal from Ω to Λ. Since T is an ideal of $\Omega = \Delta \times \Delta$, we can replace T by a smaller ideal, if necessary, so that T has the form $T = S \times S$ for some nonzero ideal S of Δ. Applying the Comparison Lemma with $M = \Lambda$ then completes the proof. ∎

5.6. Definition. We say a division R_Q-algebra D is *universally stabilizing* if cancellation holds (for *left* modules) in gen(Λ) for every R-order Λ in $D \times D$ (hence also for every order in D). When cancellation holds in gen(Λ) one says that *locally free cancellation* holds for Λ, because this implies that cancellation holds in the genus of every locally free left Λ-module. (See, e.g., [GL, 1.2 and 1.3].)

We do not know whether locally free cancellation for every R-order in $D \times D$ is equivalent to the weaker-looking condition that locally free cancellation holds for every order in D itself. However, if D is universally stabilizing, then locally free cancellation holds for every order in D^n ($\forall n \geqslant 1$), by the Cancellation Theorem below.

To check that D is universally stabilizing, it suffices to find an R-order Δ in D such that locally free cancellation holds for every R-suborder Λ of $\Delta \times \Delta$, by statement (5.6.1) below. Moreover, Lemma 5.5 gives necessary and sufficient conditions for locally free cancellation to hold for every such Λ, and we use these conditions in what follows.

(5.6.1) *Let Ω be an R-order in the semisimple artinian ring $A = \Omega_Q$, and suppose that locally free cancellation holds for every R-suborder of Ω. Then locally free cancellation holds for every R-order in A.*

Proof. Let Λ be an R-order in A. By hypothesis, locally free cancellation holds for the R-suborder $\Lambda' = \Lambda \cap \Omega$ of Ω. By the Comparison Lemma, locally free cancellation holds for all overorders of Λ', in particular for Λ. ∎

In stating our first main result, we return to the general situation in Notation 1.1: Λ is a module-finite R-algebra and R has Krull dimension $\leqslant 1$.

5.7. CANCELLATION THEOREM. *If a (left) Λ-module M satisfies the following condition, then cancellation holds in* gen(M).

(5.7.1) *For every indecomposable Λ_Q-module Y (necessarily of finite length) that occurs exactly once as a direct summand of M_Q, the division R_Q-algebra $E(Y)/\mathrm{rad}(E(Y))$ is universally stabilizing.*

Proof. By Lemma 1.4 we may suppose that Λ is an R-order in the semisimple artinian ring Λ_Q and $M = \Lambda$. Let $S_1, \ldots, S_n$ be representatives of the distinct isomorphism classes of simple left Λ_Q-modules.

By Theorem 2.8 there is an R-overorder Ω $(\Lambda \subseteq \Omega \subseteq \Lambda_Q)$ such that $\Omega = \bigoplus_i \Omega_i$ with each Ω_i an R-order in the simple artinian ring $(\Omega_i)_Q$, such that S_i is the simple left $(\Omega_i)_Q$-module, and such that each Ω_i is a direct sum of left ideals with the following properties.

$$(5.7.2) \qquad \Omega_i = \bigoplus_{h=1}^{m(i)} M_{ih}, \qquad \text{where} \quad M_{ih} \in \mathrm{gen}(M_{i1}) \text{ and } (M_{ih})_Q \cong S_i \ (\forall h).$$

Let T be a 2-sided ideal of Ω contained in Λ such that the R-modules Λ/T and Ω/T have finite length. Also, let $T_i = T\Omega_i$.

We prove the theorem by verifying the conditions of the Comparison Lemma, with $M = \Lambda$.

(i) Here it suffices to verify that cancellation holds in gen(Ω_i) for each i. So choose an i, and let $m = m(i)$. By (5.7.1) we either have that $m \geqslant 2$ or $E(S_i)$ is universally stabilizing. First assume that $m \geqslant 2$, and let

$$(5.7.3) \qquad H \oplus X \cong K \oplus X \qquad \text{with } H, K, X \text{ in gen}(\Omega_i).$$

In (5.7.2), let $L = M_{i1}$. Since Ω is a module-finite algebra over a noetherian ring of Krull dimension 1, every Ω-module in gen(L^m) is isomorphic to

$L^{m-1} \oplus L'$ for some $L' \in \text{gen}(L)$. This slight variation of Serre's direct-summand theorem is proved in [GL, 1.2]. Applying this to the modules in (5.7.3) we get

$$(5.7.4) \quad (L^{m-1} \oplus H') \oplus (L^{m-1} \oplus X') \cong (L^{m-1} \oplus K') \oplus (L^{m-1} \oplus X')$$

for suitable Λ-modules H', X', K' in $\text{gen}(L)$. We can now use the variation of Bass's cancellation theorem given in (5.2.1), and the hypothesis that $m \geqslant 2$, to conclude that $L \oplus H' \cong L \oplus K'$. Adding L^{m-2} to both sides then shows that $H \cong K$, completing the proof of (i) when $m \geqslant 2$.

We can now suppose that $m = 1$ and $E(S_i)$ is universally stabilizing. Then cancellation holds in the genus of the R-order $\Omega_i \times \Omega_i$ in $E(S_i) \times E(S_i)$, hence in the genus of Ω_i itself.

This completes the proof of (i).

(ii) In the notation of (5.7.2), note that each $M_{ih}/TM_{ih} \cong M_{i1}/TM_{i1}$ as Ω-modules, since $M_{ih} \in \text{gen}(M_{i1})$ and the R-module Ω/T has finite length. Therefore we can make the identification

$$(5.7.5) \qquad \Omega_i/T_i = \Theta_{m \times m}, \qquad \text{where} \quad \Theta = E(M_{i1}/TM_{i1}),$$

where $\Theta_{m \times m}$ denotes the ring of $m \times m$ matrices over Θ.

Let $N \in \text{gen}(\Lambda)$. By the variation of Serre's direct-summand theorem mentioned above (5.7.4), $\Omega_i N$ has a decomposition of the following form.

$$(5.7.6) \qquad \Omega_i N = \bigoplus_{h=1}^{m(i)} N_{ih}, \qquad \text{where each} \quad N_{ih} \in \text{gen}(M_{i1}).$$

Hence $(N_{ih})_Q \cong S_i$ so we can also make the identification $E(\Omega_i N/T_i N) = \Theta_{m \times m}$.

Now let $\vartheta \in E^*(\Omega N/TN)$, and suppose $\nu(\vartheta) = 1$ in $\mathbf{K}_1(\Omega N/TN)$. Then $\vartheta = (\vartheta_1, ..., \vartheta_n)$ with each $\vartheta_i \in E^*(\Omega_i N/T_i N)$ and $\nu(\vartheta_i) = 1$ in $\mathbf{K}_1(E(\Omega_i N/T_i N)) = \mathbf{K}_1(\Theta)$.

Fix an index i, and write $m = m(i)$. We consider the two possibilities in (5.7.1) separately.

First consider the possibility $m \geqslant 2$. Since $\nu(\vartheta_i) = 1$ in $\mathbf{K}_1(E(\Omega_i N/T_i N)) = \mathbf{K}_1(\Theta)$, since $m \geqslant 2$, and since the artinian ring $E(\Omega_i N/T_i N)$ has 1 in its stable range, we conclude that ϑ_i is a product of elementary matrices. Therefore, by Corollary 3.5, ϑ_i belongs to the image of $E^*(\Omega_i N)$ in $E^*(\Omega_i N/T_i N)$.

On the other hand, suppose that $m = 1$ and $E(S_i)$ is universally stabilizing. Then by Lemma 5.5, applied to the order $\Delta = E(\Omega_i N)$ in $D = E(S_i)$, ϑ_i again belongs to the image of $E^*(\Omega_i N)$ in $E^*(\Omega_i N/T_i N)$.

Therefore, in either situation, $\vartheta \in E^*(\beta N) \cdot E^*(\Omega N) = \lambda(N)$, completing the proof of (ii).

(iii) Let $N \in \mathrm{gen}(M)$ and $\varphi \in GL_2(E(\Omega N))$. We have $\varphi = (\varphi_1, ..., \varphi_n)$ with each $\varphi_i \in GL_2(E(\Omega_i N))$. To show that $\nu(\varphi) \in \nu\lambda(N)$ it suffices to show that each $\nu(\varphi_i)$ belongs to the image of $E^*(\Omega_i N)$ in $\mathbf{K}_1(E^*(\Omega_i N/T_i N))$.

We can therefore fix i and then simplify the notation by writing $\varphi = \varphi_i$ and $m = m(i)$. By (5.7.1) there are two cases to consider.

First suppose $m \geqslant 2$, and let $E = \Omega\text{-end}(\Omega_i N)$ (Ω-endomorphisms, written as right operators). By (5.7.6), E is a direct sum of left ideals:

$$(5.7.7) \qquad E = \bigoplus_{h=1}^{m(i)} Y_h, \qquad \text{where} \quad Y_h = \Omega\text{-hom}(N_{ih}, \Omega_i N).$$

We have $E_Q \cong (\Omega_i)_Q$ as rings, and $(Y_i)_Q \cong S_i$ as $(\Omega_i)_Q$-modules. Moreover, every $Y_h \in \mathrm{gen}(Y_1)$. Let $\Delta = \Omega\text{-end}(Y_1)$, written as right operators. By Lemma 2.9, $E \cong \Delta\text{-end}(Y_1)$ and L_1 is a locally free right Δ-module of constant rank. Hence, by Lemma 5.1, both $GL_2(E(\Omega_i N))$ and $E^*(\Omega_i N)$ have the same image in $\mathbf{K}_1(E^*(\Omega_i N))$, hence in $\mathbf{K}_1(E^*(\Omega_i N/T_i N))$, as desired.

Suppose next that $m = 1$ and $E(S_i)$ is universally stabilizing. Again we apply Lemma 5.5, applies to the order $\Delta = E(\Omega_i N)$ in $D = E(S_i)$. If $\varphi \in GL_2(E(\Omega_i N))$, then $\nu(\varphi) \in \nu E^*(\Omega_i N)$ so $\nu(\varphi) \in \nu\lambda(N)$, as desired. ∎

5.8. Lemma. *Every commutative division R_Q-algebra is universally stabilizing.*

Proof. Let the given division algebra be D. We verify the conditions of Lemma 5.5 for every R-order Δ in D.

Condition (i). Suppose that (5.7.3) holds with $\Omega_i = \Delta$. Since D is a field, the order Δ in D is a noetherian integral domain. The Δ-modules H, K, X are projective of rank 1, hence can be taken to be invertible ideals of Δ. We are now in a situation in which cancellation is known to hold. For example, taking the second exterior power in (5.7.3) gives $HX \cong KX$; and multiplying by X^{-1} then shows that $H \cong K$.

Condition (ii). Here $\vartheta = 1$, hence belongs to the image of Δ^* in $E^*(\Delta/S)$.

Condition (iii). If $\varphi \in GL_2(\Delta)$ has determinant d, then $\varphi = \mathrm{diag}(d, 1) \cdot \beta$ where $\det(\beta) = 1$. Since 1 is in the stable range of the artinian ring (Δ/S), β is a product of elementary matrices, so we have $\nu(\varphi) = \nu(d) \in \nu(\Delta^*)$, as desired.

An immediate consequence of the preceding lemma and Cancellation Theorem 5.7 is:

5.9. Corollary (Drozd Cancellation Theorem). *If a (left) Λ-module M satisfies the following "Drozd Condition," then cancellation holds in* $\mathrm{gen}(M)$.

(5.9.1) *For every indecomposable Λ_Q-module Y (necessarily of finite length) that occurs exactly once as a direct summand of M_Q, the division R_Q-algebra $E(Y)/\mathrm{rad}(E(Y))$ is commutative.*

In Drozd's original result [Dz], R was a Dedekind domain, Λ was an R-order in a semisimple separable algebra A over the field of fractions of R, and M was a Λ-lattice. This was extended in two ways, in [G] and then [GL]. First, M was allowed to be an arbitrary (finitely generated) Λ-module. Second, R was allowed to be any reduced noetherian ring of dimension $\leqslant 1$, but Λ was assumed to be contained in a maximal R-order in A. The key ingredient in the present, more general and more natural version of the theorem is dropping the hypothesis that Λ be contained in a maximal order. Once that is done, allowing nilpotent ideals in Λ and R is relatively straightforward.

Next we state some conditions on Λ under which cancellation holds in every genus of Λ-modules. We state part (i) of the following corollary separately because it makes a nice application of our results to commutative rings.

We say that a division ring is *associated with* a semisimple artinian ring A if it is one of the division rings that occur when A is expressed as a direct product of full matrix rings over division rings.

5.10. COROLLARY. *Suppose that*

(i) *Λ is any commutative reduced noetherian ring of dimension* 1; *or, more generally,*

(ii) *Λ is a semiprime (has no nonzero nilpotent ideals) R-algebra and every division R-algebra associated with the semisimple artinian ring Λ_Q is universally stabilizing.*

Then cancellation holds in every genus of Λ-modules.

Proof. By Lemma 5.8, situation (i) is the special case of situation (ii) in which $\Lambda = R$. Thus we only have to prove (ii). In view of the Drozd Cancellation Theorem, it suffices to prove that, for every indecomposable Λ_Q-module Y, $E(Y)$ is universally stabilizing.

Since Λ_Q is a semisimple artinian ring, every indecomposable Λ_Q-module Y is a simple module and $E(Y)$ is one of the division R-algebras (universally stabilizing, by hypothesis) associated with Λ_Q. ∎

5.11. *Remarks.* (i) The hypotheses of the preceding corollary are satisfied if Λ_Q is a direct product of full matrix rings over fields.

(ii) (Jacobinski Cancellation Theorem) The hypotheses of the preceding corollary are also satisfied if R_Q is a global field and all division algebras of Λ_Q satisfy the Eichler Condition, as defined in [S1] or [R]. (See the next paragraph.) This generalization of Jacobinski's Cancellation theorem was proved in [G1, 3.8].

It suffices to check that all such division algebras are universally stabilizing, a well-known theorem (but stated in different terminology) in integral representation theory. [CR2, (51.28)] or [S1].

(iii) The preceding corollary becomes false if Λ contains nilpotent ideals, even if Λ is commutative and its underlying abelian group is free of finite rank, as we show in [GLW].

Our final result gives another sense in which cancellation in a genus is a type of stability.

5.12. THEOREM. *Let M and M' be Λ-modules, and suppose either*

(i) *Cancellation holds in* $\operatorname{gen}(M)$ *and* $\operatorname{gen}(M')$; *or*

(ii) *Cancellation holds in* $\operatorname{gen}(M)$ *and every indecomposable direct summand of $(M')_Q$ is isomorphic to a direct summand of M_Q. Then cancellation holds in* $\operatorname{gen}(M \oplus M')$.

Before beginning the proof, we note that, for orders in semisimple artinian rings, there is an essentially equivalent reformulation of situation (ii). A version of this was proved by Gruenberg and Linell [GrL, 1.3] for lattices over the orders that occur in integral representation theory.

5.13. COROLLARY. *Let M be a Λ-module, where Λ is an R-order in the semisimple artinian ring Λ_Q. Suppose that cancellation holds in* $\operatorname{gen}(M)$ *and M is a faithful Λ-module. Then cancellation holds in* $\operatorname{gen}(M \oplus M')$ *for every Λ-module M'.*

5.14. *Proof of Theorem* 5.12. By Lemma 1.4 we can suppose that Λ is an R-order in the semisimple artinian ring Λ_Q and $\Lambda = M \oplus M'$. Let $S_1, \ldots, S_n$ be representatives of the distinct isomorphism classes of simple left Λ_Q-modules.

We carry out the proof of the theorem in situation (i), and intersperse comments on what modifications are necessary for situation (ii).

By Theorem 2.8 there is an R-overorder Ω $(\Lambda \subseteq \Omega \subseteq \Lambda_Q)$ such that $\Omega = \bigoplus_i \Omega_i$ where each Ω_i is an R-order in the simple artinian ring $(\Omega_i)_Q$, and S_i is the simple left $(\Omega_i)_Q$-module.

After a second application of Theorem 2.8 we can replace Ω by a larger R-order in Λ_Q such that each left Ω-module $\Omega_i M$ has a decomposition of the form

$$\Omega_i M = \bigoplus_{h=1}^{m(i)} M_{ih}, \tag{5.14.1}$$

where

$$M_{ih} \in \text{gen}(M_{i1}) \text{ and } (M_{ih})_Q \cong S_i \ (\forall h)$$

and each $\Omega_i M'$ has an analogous decomposition that we refer to as (5.14.1)′. We write $m(i)'$ for the number, in decomposition (5.14.1)′ that corresponds to $m(i)$ in (5.14.1).

For some values of i, we can have $m(i)=0$ or $m(i)'=0$. However, at least one of $m(i)$ and $m(i)'$ is always nonzero since $M \oplus M' = \Lambda$, hence $\Omega_i M \oplus \Omega_i M' = \Omega_i$. Note that, in situation (ii), we always have $m(i) \neq 0$.

Let T be a 2-sided ideal of Ω contained in Λ such that the R-modules Λ/T and Ω/T have finite length; and let $T_i = T\Omega_i$.

In situation (i) cancellation holds in both gen(M) and gen(M'), so the conditions of the Comparison Lemma are satisfied by M and M'. We prove the theorem by showing that these conditions are also satisfied by $M \oplus M' = \Lambda$. In situation (ii) the conditions of the Comparison Lemma are only satisfied by M.

Condition (i). We want to show that cancellation holds in gen($\Omega(M \oplus M')$), which is equivalent to cancellation holding in gen($\Omega_i(M \oplus M')$) for every i.

Choose an index i, and let $m = m(i)$ and $m' = m(i)'$. If $m + m' \geqslant 2$, then cancellation holds in the genus of $\Omega_i(M \oplus M') = \Omega_i$ by the Drozd Cancellation Theorem.

The remaining possibility is $m = 1$ and $m' = 0$, or vice versa. In situation (i) we can assume, by symmetry, that $m = 1$; and in situation (ii) we necessarily have $m = 1$.

Since cancellation holds in gen(M), the Comparison Lemma shows that cancellation holds in gen(ΩM), hence in gen($\Omega_i M$) which equals gen($\Omega_i(M \oplus M')$) since $\Omega_i M' = 0$. Thus condition (i) is proved.

Condition (ii). By Corollary 4.3, every module in gen($M \oplus M'$) is isomorphic to $N \oplus N'$ for some $N \in \text{gen}(M)$ and $N' \in \text{gen}(M')$.

At this point it is necessary to visualize $\lambda(N)$, $\lambda(N')$, and $\lambda(N \oplus N')$ appropriately. (See Notation 5.3.) We assume that pullback diagrams analogous to the second diagram in (3.1.1) have been formed for N and N', and we use the direct sum of these diagrams as the pullback diagram for $N \oplus N'$. By Corollary 4.3, N and N' have decompositions whose terms lie in the same genera as the corresponding terms as those in (5.14.1) and (5.14.1)′. This yields a decomposition

$$\Omega_i N \oplus \Omega_i N' = \left(\bigoplus_{h=1}^{m(i)} N_{ih} \right) \oplus \left(\bigoplus_{h=1}^{m(i)'} (N_{ih})' \right) \tag{5.14.2}$$

with each $N_{ih} \in \text{gen}(M_{i1})$ and $(N_{ih})' \in \text{gen}(M_{i1})'$. We view each element of

$E(\Omega_i N \oplus \Omega_i N')$ as an $m(i)+m(i)'$ matrix, acting on the right, whose entries are homomorphisms between the coordinate modules in (5.14.2.)

We define a multiplicative embedding of $E^*(\Omega_i N)$ and $E^*(\Omega_i N')$ in $E^*(\Omega_i N \oplus \Omega_i N')$ by making the identifications

$$\alpha = \mathrm{diag}(\alpha, 1) \qquad \text{and} \qquad \alpha' = \mathrm{diag}(1, \alpha'), \tag{5.14.3}$$

where $\alpha \in E^*(\Omega_i N)$, $\alpha' \in E^*(\Omega_i N')$, and "1" is used to represent identity matrices of appropriate sizes. This yields (coordinatewise) multiplicative embeddings of $E^*(\Omega N)$ and $E^*(\Omega N')$ in $E^*(\Omega N \oplus \Omega N')$, as well as corresponding embeddings for the corresponding subgroups of $E^*(\Omega(N \oplus N')/T(N \oplus N'))$.

Note that $E^*(\beta N)$ and $E^*(\beta N')$ commute with each other, because of the 1's in (5.14.3), and

$$E^*(\beta N') \cdot E^*(\beta N) \subseteq E^*(\beta(N \oplus N')) \tag{5.14.4}$$

because of the 1's in (5.14.3). Next note that the elements of $\lambda(N)$ commute with those of $\lambda(N')$, again because of these 1's.

Since cancellation holds in gen(N), the Comparison Lemma shows that $\lambda(N)$ is a normal subgroup of $E^*(\Omega N/TN)$. In situation (i), a similar statement holds for N'. In situation (ii), $\lambda(N')$ might not be a group. We claim that, in either situation,

$$\lambda(N) \cdot \lambda(N') \subseteq \lambda(N \oplus N'). \tag{5.14.5}$$

Since $\lambda(N)$ centralizes $E^*(\beta N')$ we have

$$\begin{aligned}\lambda(N) \cdot \lambda(N') &= \lambda(N) \cdot E^*(\beta N') \cdot E^*(\Omega N') = E^*(\beta N') \cdot \lambda(N) \cdot E^*(\Omega N') \\ &= E^*(\beta N') \cdot E^*(\beta N) \cdot E^*(\Omega N) \cdot E^*(\Omega N')\end{aligned}$$

so the proof of (5.14.5) is completed by (5.14.4).

To verify condition (ii) of the Comparison Lemma for $N \oplus N'$ take any $\vartheta \in \bar{E}^* = E^*(\Omega(N \oplus N')/T(N \oplus N'))$ such that $\nu(\vartheta) = 1$ in $\mathbf{K}_1(\bar{E})$. We can write $\vartheta = (\vartheta_1, \vartheta_2, \ldots)$ with each $\vartheta_i \in (\bar{E}^*)_i = E^*(\Omega_i(N \oplus N')/T_i(N \oplus N'))$. Then $\nu(\vartheta_i) = 1$ in $\mathbf{K}_1(\bar{E}^*)_i$ for each i.

We call an automorphism ε in $\bar{E}^*$ *elementary* if each component matrix ε_i of ε is an elementary matrix in $(\bar{E}^*)_i$.

We want to show that $\vartheta \in \lambda(N \oplus N') = E^*(\beta(N \oplus N')) \cdot E^*(\Omega(N \oplus N'))$. Recall, from Corollary 3.5, that all elementary automorphisms belong to the image of $E^*(\Omega(N \oplus N'))$ in $\bar{E}^*$. Therefore, in proving that $\vartheta \in \lambda(N \oplus N')$, we can replace ϑ by $\vartheta\varepsilon$ for any elementary automorphism ε.

Since artinian rings have 1 in their stable range, a suitable number of replacements of this form enables us to transform each ϑ_i into a diagonal matrix with at most one diagonal entry $\delta_i \neq 1$. Moreover, we can put δ_i anywhere we wish on the main diagonal of ϑ_i. Therefore we now have each $\vartheta_i \in E^*(\Omega N/TN)$ or $E^*(\Omega N'/TN')$.

In situation (i), where cancellation holds in both gen(M) and gen(M'), the Comparison Lemma shows that each ϑ_i belongs to $\lambda(N)$ or $\lambda(N')$ and both $\lambda(N)$ and $\lambda(N')$ are groups. Therefore (5.14.5) shows that $\vartheta \in \lambda(N \oplus N')$, as desired.

In situation (ii), we have $\Omega_i N \neq 0$ for every i. So, when diagonalizing each ϑ_i we put that δ_i into the (1, 1)-entry of ϑ_i. The Comparison Lemma shows that $\lambda(N)$ is a group, so we now show that $\vartheta \in \lambda(N \oplus N')$ just as in situation (i). This concludes the proof of condition (ii).

Condition (iii). Since $\mathbf{K}_1(\bar{E})$ is an abelian group, $\nu\lambda(N \oplus N')$ is again a group. Hence it suffices to show that for each i, and for each $\vartheta_i \in GL_2(E(\Omega_i(N \oplus N')))$, we have $\nu(\vartheta_i) \in \nu\lambda(N \oplus N')$.

Fix i and let $m = m(i)$ and $m' = m(i)'$ in decomposition (5.14.2). We consider two cases.

First assume that $m + m' \geqslant 2$. Exactly as in the proof of Cancellation Theorem 5.7, but with $N \oplus N'$ in place of N and "$m + m' \geqslant 2$" in place of "$m \geqslant 2$," we show that $\nu(\vartheta_i) \in \nu\lambda(N \oplus N')$.

The remaining case is $m = 1$, $m' = 0$ or vice versa. In situation (i) we can assume, by symmetry, that $m = 1$; and in situation (ii) we necessarily have $m = 1$. Thus, in either case, cancellation holds in gen(N). The Comparison Lemma shows that $\nu(\vartheta_i) \in \nu\lambda(N)$ which is contained in $\nu\lambda(N \oplus N')$ by the proof of (5.14.5) since $\mathbf{K}_1$ is abelian. ▌

5.15. *Remark.* A more general form of cancellation is often used in stating Bass's cancellation theorem, namely:

(5.15.1) $M \oplus X \cong N \oplus X$ implies $M \cong N$ whenever X is a direct summand of M^n, for some n

(provided suitable local hypotheses hold). If cancellation holds in gen(M), then (5.15.1) also holds. For, by adding a suitable direct summand, we may assume that $X = M^n$ for some n. By local cancellation, we have $N \in$ gen(M), so $M \oplus M \cong N \oplus M$ by (5.2.1), hence $M \cong N$ by cancellation in gen(M).

6. Roiter's Theorem

Recall that $X \mid M$ denotes that X is isomorphic to a direct summand of M. Our first lemma extends [G3, 3.1]. For esthetic reasons, we note that condition (ii), below, is equivalent to $X_\mathfrak{p} \mid N_\mathfrak{p}$ ($\forall$ minimal prime ideals $\mathfrak{p}$ of R).

6.1. LEMMA. *Let X, M, N be Λ-modules. Suppose*:

(i) $X_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ ($\forall$ *maximal ideals* $\mathfrak{m}$ *of* R); *and*

(ii) $X_Q \mid N_Q$.

Then $X \mid M \oplus N$.

Proof. It suffices to find Λ-homomorphisms $\vartheta: M \to X$ and $\varphi: N \to X$ such that, for every $\mathfrak{m}$, either $\vartheta_{\mathfrak{m}}$ or $\varphi_{\mathfrak{m}}$ is a split surjection. For then $[\vartheta, \varphi]: M \oplus N \to X$ is locally a split surjection at all maximal ideals, hence is globally a split surjection.

By (ii) and Lemma 2.1(ii), there exists $\varphi: N \to X$ such that $\varphi_{\mathfrak{m}}$ is a split surjection (a $\forall \mathfrak{m}$). Then by (i) and the Chinese Remainder Theorem there exists $\vartheta: M \to X$ such that $\vartheta_{\mathfrak{m}}$ is a split surjection at the finite number of $\mathfrak{m}$ at which $\varphi_{\mathfrak{m}}$ is not. ∎

We now weaken hypothesis (ii) of the preceding lemma.

6.2. THEOREM ("Roiter's Theorem"). *Let X, M, N be Λ-modules. Suppose*:

(i) $X_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ ($\forall$ *maximal ideals* $\mathfrak{m}$ *of* R); *and*

(ii) *every indecomposable direct summand of X_Q is isomorphic to a summand of N_Q.*

Then $X \mid M \oplus N$.

Proof. We can assume that Λ is an order in a semisimple artinian ring Λ_Q, and X, M, N are Λ-lattices, by Lemmas 1.4 and 1.6 (applied to $M \oplus N$), and the modification of Lemma 1.6(iv) that replaces $\mathfrak{m}$ with Q.

By (i) and Theorem 4.1 we have $M \cong X' \oplus V$ for some $X' \in \operatorname{gen}(X)$ and some V. So it suffices to show that $X \mid X' \oplus N$.

By (ii) there is a positive integer n such that $X_Q \mid (N_Q)^n$. So by the preceding lemma, $X \mid X' \oplus N^n$. Say

$$X' \oplus N^n \cong X \oplus Y. \tag{6.2.1}$$

By local cancellation we get $Y \in \operatorname{gen}(N^n)$. So by [GL, 1.2], $Y \cong N' \oplus N^{n-1}$ with $N' \in \operatorname{gen}(N)$. Substituting this into (6.2.1) gives

$$X' \oplus N^n \cong X \oplus N' \oplus N^{n-1}. \tag{6.2.2}$$

Thus it now suffices to cancel N^{n-1} from (6.2.2).

Adding X^{n-1} to both sides. We get

$$(X' \oplus N) \oplus (X \oplus N)^{n-1} \cong (X \oplus N') \oplus (X \oplus N)^{n-1} \tag{6.2.3}$$

so it now suffices to show that cancellation holds in the genus of $X \oplus N$.

Suppose first that every isomorphism class of indecomposable direct summand of N_Q appears at least once as a direct summand of X_Q. Then, when $(X \oplus N)_Q$ is written as a direct sum of indecomposable modules, every isomorphism class of indecomposable direct summand occurs at least twice. So the hypothesis (5.7.1) of our Cancellation Theorem is vacuously satisfied, and we are done.

Note that we have not yet made use of the reduction to the case that Λ_Q is semisimple artinian and M, N, X are Λ-lattices.

In the general case, choose a Λ-lattice C such that C_Q is the direct sum of one copy of each indecomposable direct summand of N_Q that does not appear in X_Q. After adding C^n to (6.2.3), the preceding argument shows

$$X' \oplus C \oplus N \cong X \oplus C \oplus N'. \tag{6.2.4}$$

To show that $X \mid X' \oplus N$, thus completing the proof, it suffices to show that the projection of X in the C on the left-hand side of (6.2.4) equals 0; and this follows if we show that $\hom(X, C) = 0$.

Since Λ_Q is semisimple artinian and X and C are Λ-lattices, it suffices to show that $\hom(X_Q, C_Q) = 0$, and this is true by our choice of C. ▮

6.3. COROLLARY. *Let Λ be an R-order in a semisimple artinian ring A. Suppose X, M, N are Λ-modules such that $X_{\mathfrak{m}} \mid M_{\mathfrak{m}}$ ($\forall \mathfrak{m}$) and N is faithful. Then $X \mid M \oplus N$.*

Proof. Since N is a faithful Λ-module, N_Q is a faithful Λ_Q-module. Since the ring Λ_Q is semisimple artinian, this implies that *every* indecomposable Λ_Q-module is isomorphic to a direct summand of N_Q. Thus condition (ii) of the preceding theorem is satisfied, and the desired conclusion follows. ▮

The above corollary was proved by Guralnick [G3, 6.4], under the additional hypotheses that R is a Dedekind domain and Λ is contained in a maximal order. This, in turn, generalized Roiter's original result [Ro], which assumed, in addition, that the residue fields of R are finite and X, M, N are Λ-lattices.

A variant of this corollary was proved by Jacobinski [J], for lattices over the orders studied by Roiter, and then generalized by Guralnick [G3, 6.3] to modules over the rings he worked with in the foregoing paragraph. Our version of this variant is:

6.4. COROLLARY ("Jacobinski's Theorem"). *Let X and V be Λ-modules. Suppose*:

(i) $X_{\mathfrak{m}} \mid V_{\mathfrak{m}}$ (*$\forall$ maximal ideals $\mathfrak{m}$ of R*); *and*

(ii) *every indecomposable summand of X_Q occurs strictly more often in a decomposition of V_Q.*

Then $X \mid V$.

Proof. By Theorem 4.1 we have $V = X' \oplus N$ with $X' \in \operatorname{gen}(X)$. Since the Krull–Schmidt theorem holds for modules over the artinian ring Λ_Q, condition (ii) implies condition (ii) of Theorem 6.2. Now use Theorem 6.2. ∎

6.5. EXAMPLES. The need for the additional summand N in Theorem 6.2 is illustrated by the well-known example in which Λ is a Dedekind domain, $X = \Lambda$, and M is any nonprincipal ideal of Λ.

For an example illustrating more of the intricacies of the preceding theorems, let $f_i: R_i \twoheadrightarrow K$ $(i = 1, 2)$ be ring homomorphisms of Dedekind domains R_i onto a field K. Then let $R = \Lambda = \{(x_1, x_2) \in R_1 \times R_2 \mid f_1(x_1) = f_2(x_2)\}$. Also, let H_i be a nonprincipal ideal of each R_i with H_i prime to $\ker(f_i)$ [so that f_i maps H_i onto K] and let $H = \{(h_1, h_2) \in H_1 \times H_2 \mid f_1(h_1) = f_2(h_2)\}$.

Then H is a Λ-module. Λ is clearly not isomorphic to a direct summand of $H \oplus (R_1 \times 0)$. This illustrates the need for the faithfulness of N in Corollary 6.3 and the fact that every composition factor must occur "strictly more often" in Corollary 6.4. On the other hand, either of these corollaries shows that Λ is a direct summand of $H \oplus (R_1 \times R_2)$.

The way in which Roiter's theorem is stronger than Serre's theorem is illustrated by the fact that H is a locally free projective Λ-module, but $(R_1 \times R_2)$ is not projective. The proofs of these assertions are a slight variation of [L, 2.4 and 5.6].

REFERENCES

[B] H. BASS, "Algebraic *K*-Theory," Benjamin, New York, 1968.

[CR1] C. W. CURTIS AND I. REINER, "Methods of Representation Theory, with Applications to Finite Groups and Orders," I, Wiley–Interscience, New York, 1981.

[CR2] C. W. CURTIS AND I. REINER, "Methods of Representation Theory, with Applications to Finite Groups and Orders," II, Wiley–Interscience, New York, 1987.

[CRo] A. W. CHATTERS AND J. C. ROBSON, Decomposition of orders in semiprimary rings, *Comm. Algebra* **8** (1980), 517–532.

[Dz] A. DROZD, Adeles and integral representations, *Izv. Akad. Nauk SSSR* **33** (1969), 1019–1026.

[E] D. EISENBUD, Subrings of artinian and noetherian rings, *Math. Ann.* **185** (1970), 247–249.

[EG] D. R. ESTES AND R. M. GURALNICK, Module equivalences: Local to global when primitive polynomials represent units, *J. Algebra* **77** (1982), 138–157.

[F] A. FRÖHLICH, Locally free modules over arithmetic orders, *J. Reine Angew. Math.* **274/275** (1975), 112–124.

[GrL] K. W. GRUENBERG AND P. A. LINELL, Minimal resolutions of lattices over finite groups, *Illinois J. Math.* **32** (1988), 361–374.

[G1] R. M. GURALNICK, The genus of a module, *J. Algebra* **18** (1984), 169–177.

[G2] R. M. GURALNICK, Power cancellation of modules, *Pacific J. Math.* **124** (1986), 131–144.

[G3] R. M. GURALNICK, The genus of a module. II. Roiter's theorem, power cancellation, and extension of scalars, *J. Number Theory* **26** (1987), 149–165.

[GL] R. M. GURALNICK AND L. S. LEVY, Presentations of modules when ideals need not be principal, *Illinois J. Math.* **32** (1988), 593–653.

[GLW] R. M. GURALNICK, L. S. LEVY, AND R. WARFIELD, Cancellation counterexamples in Krull dimension 1, *Proc. Amer. Math. Soc.* **109** (1990), 323–326.

[GW] K. R. GOODEARL AND R. B. WARFIELD, Algebras over zero-dimensional rings, *Math. Ann.* **223** (1976), 157–168.

[H] M. HOCHSTER, Non-openness of loci in Noetherian rings, *Duke Math. J.* **40** (1973), 215–219.

[J] H. JACOBINSKI, Genera and decomposition of lattices over orders, *Acta Math.* **121** (1968), 1–29.

[L] L. S. LEVY, Krull–Schmidt uniqueness fails dramatically over subrings of $Z \oplus Z \oplus \cdots \oplus Z$, *Rocky Mountain J. Math.* **13** (1983), 659–678.

[N] M. NAGATA, "Local Rings," Interscience, New York, 1962.

[R] I. REINER, "Maximal Orders," Academic Press, New York, 1975.

[Ro] A. ROITER, On integral representations belonging to a genus, *Izv. Akad. Nauk SSSR Ser. Mat.* **30** (1966), 1315–1324; English translation, *Amer. Math. Soc. Transl. (2)* **71** (1968), 49–59.

[Ru] D. RUSH, Torsionfree modules over one-dimensional rings, *J. Pure Appl. Algebra* **52** (1988), 133–146.

[S1] R. G. SWAN, Strong approximation and locally free modules, *in* "Ring Theory and Algebra" (B. McDonald, Ed.), Vol. III, pp. 153–224, Dekker, New York, 1980.

[S2] R. G. SWAN, Wiegand's patching method over orders, unpublished notes.

[W] R. WIEGAND, Cancellation over commutative rings of dimension one and two, *J. Algebra* **88** (1984), 438–459.

[WW] R. WIEGAND AND S. WIEGAND, Stable isomorphism of modules over one-dimensional rings, *J. Algebra* **107** (1987), 425–435.